Meaning and interpretation in mathematics

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Colin McLarty, 2013

Fermat's Last Theorem is just about numbers, so it seems like we ought to be able to prove it by just talking about numbers.

Briançon-Skoda Theorem (1974)

Let R be either the formal or convergent power series ring in d variables and let I be an ideal of R. Then $\overline{I^d} \subseteq I$, where $\bar{I}$ is the *integral closure* of an ideal I.

Lipman & Teissier, "Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals", *Michigan Mathematical Journal* (1981)

The proof given by Briançon and Skoda of this completely algebraic statement is based on a quite transcendental deep result of Skoda. . . . The absence of an algebraic proof has been for algebraists something of a scandal—perhaps even an insult—and certainly a challenge.

Lipman & Tessier then give such an algebraic proof.

What is this "scandal", this "challenge"?

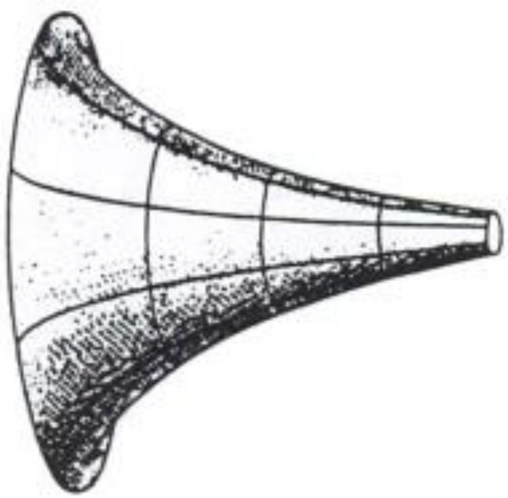
It is common nowadays to formulate the issues raised here in terms of purity of methods.

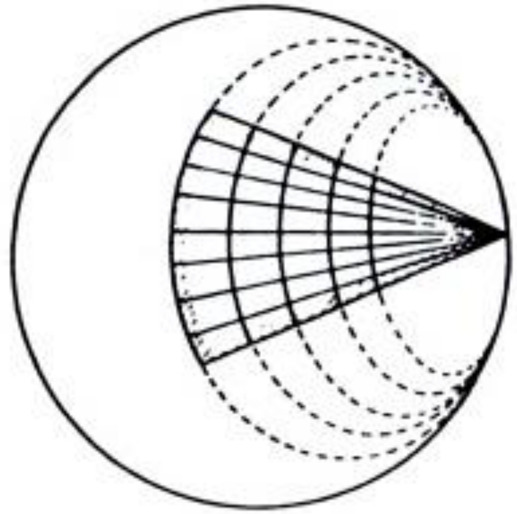
Roughly, a solution to a problem, or a proof of a theorem, is pure if it draws only on what is "close" or "intrinsic" to that problem or theorem.

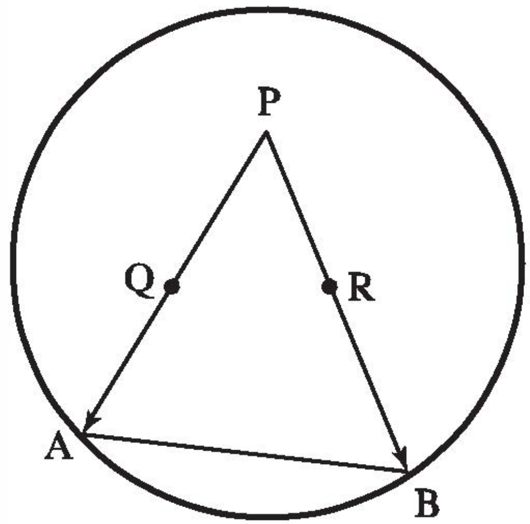
Other common language: avoids what is "extrinsic", "extraneous", "distant", "remote", "alien" or "foreign" to the problem or theorem.

Hilbert, "Lectures on Euclidean Geometry", 1898–1899

Therefore we are for the first time in a position to put into practice *a critique of means of proof*. In modern mathematics such criticism is raised very often, where the aim is to preserve *the purity of method* [*die Reinheit der Methode*], i.e. to prove theorems if possible using means that are suggested by [*nahe gelegt*] the content [*Inhalt*] of the theorem.







Poincaré, "Les géométries non euclidiennes" (1891)

Let us construct a kind of dictionary by making a double series of terms written in two columns, and corresponding each to each, just as in ordinary dictionaries the words in two languages which have the same meaning correspond to one another. Let us now take Lobachevsky's theorems and translate them by the aid of this dictionary, as we would translate a German text with the aid of a German-French dictionary. We shall then obtain the theorems of ordinary geometry.

T is interpretable in T^* if there is a way of translating the atomic formulas of T into formulas of T^* such that for the induced map $\varphi \mapsto \varphi^*$

$$\text{if } T \text{ proves } \varphi, \text{ then } T^* \text{ proves } \varphi^*.$$

One then says that two theories are mutually interpretable if each interprets the other.

The interpretation serves as a dictionary for translating statements in one theory into statements of another, in such a way that provability is preserved.

Whether a proof is pure or not, in the Hilbert, "topical" sense, comes down to whether it draws only what belongs to the content of what is being proved.

Topical purity is a matter of meaning.

An interpretation provides a dictionary for translating statements of a theory T into statements of another theory T^*, in such a way that if a statement in the language of T is provable in T, then its translation into the language of T^* is provable in T^*.

This type of translation preserves provability. But ordinarily we look to dictionaries to preserve meanings.

Is an interpretation a change in content? Do interpretations preserve meanings?

Heck, "Frege's Theorem: An Introduction", 1998

What the interpretability result establishes is just that what *look like* the axioms of Euclidean geometry can be proven within analysis. The question is whether what *looks like* the parallel postulate really does mean what the parallel postulate means.

Heck, *Frege's Theorem*, 2011

The representability of geometrical objects in real 3-space does not necessarily yield a proof of the axioms of Euclidean geometry, not if these axioms are supposed to have the same content as the axioms as we ordinarily understand them.

According to Heck, the truths of Euclidean geometry can be identified "by what they mean and not just by their orthographic or syntactic structure."

In the *Grundlagen der Geometrie* (1899), David Hilbert showed:

- Desargues' Theorem holds in a projective plane iff that plane can be coordinatized by a division ring.
- Pappus' Theorem holds in a projective plane iff that plane can be coordinatized by a field.

Hilbert showed, given a projective plane, how to construct its "algebra of segments", and that if this plane satisfies Desargues' / Pappus' theorem, multiplication in this algebra is associative / commutative.

He also showed how to recover the relevant geometries from the relevant algebraic structures.

Paying attention to the proofs of these theorems, one sees quickly that:

- The division ring axioms are mutually interpretable (with parameters) with the axioms for Desarguesian projective planes.
- The field axioms are mutually interpretable (with parameters) with the axioms for Pappian projective planes.

Thus one theory can be interpretable in a quite different theory.

One theory may be geometrical, while the other is algebraic, for example.

If interpretations preserve meanings, then statements concerning Desarguesian projective planes have the same meaning as their translations concerning division rings.

Thus a pure proof of a purely geometric theorem could draw as much on algebraic concepts as it does on geometric concepts.

This obliterates the traditional understanding of purity.

But (some) mathematicians today still recognize this traditional understanding as (sometimes) meriting attention.

That's one reason to think interpretations don't preserve meanings.

It is a commonplace to say that complex numbers can be reduced to pairs of real numbers, as is familiar since Hamilton.

One might conclude: complex numbers can be used freely in proofs of real theorems because pairs of reals add nothing new in terms of impurity.

Can we make sense of this in terms of interpretability?

Świerczkowski, "Interpretations of Euclidean Geometry" (1990)

We know since Descartes that points of the Euclidean n-dimensional space can be identified with n-tuples of real numbers. Moreover, each statement of n-dimensional geometry that involves a variable x representing a point has its counterpart in algebra that is a statement involving n variables $x_1, \ldots, x_n$ (the "coordinates" of x). In such a situation, we say... that we have an *n-dimensional interpretation* of the language of geometry in the language of algebra.

Let E_n be Tarski's theory of n-dimensional Euclidean geometry, with a ternary predicate for betweenness and a quaternary predicate for equidistance between two pairs of points.

Świerczkowski notes that the Cartesian n-dimensional interpretation also yields an interpretation of the theory E_n in RCF, the theory of real closed fields.

The notion of "n-dimensional interpretation" seems to have been introduced by Szczerba in 1975 ("Interpretations of Elementary Theories"), in the context of understanding interpretations between different axiomatizations of elementary geometry.

Montague's model-theoretic characterization of interpretability ("Interpretability in Terms of Models", 1965) was extended to n-dimensional interpretations by van Bentham and Pearce ("A Mathematical Characterization of Interpretation Between Theories", 1984).

When a theory S is n-dimensional interpretable in a theory T, we write $S \prec^n T$.

Thus $E_n \prec^n RCF$.

For example, $E_2 \prec^2 RCF$ by interpreting points in the Euclidean plane as Cartesian coordinates (that is, pairs) in RCF.

Mycielski asked if there is a more economical way of interpreting E_n in RCF than the Cartesian way, with k-tuples of variables for $k < n$, and conjectured that there isn't.

Theorem (Świerczkowski). For $k < n$, $E_n \not\rhd^k RCF$.

The proof generalizes Boffa's proof for $n = 2$ and $k = 1$ (1980) using a result on triangulation of semi-algebraic sets proved, independently, by Łojasiewicz (1964), Hironaka (1975), Coste (1982), and van den Dries (1985).

The study of n-dimensional interpretations has recently been advanced by Daniel Alscher (*Theorien der reellen Zahlen und Interpretierbarkeit*, 2015).

Theorem (Alscher). $ACF_0 \prec^2 RCF$.

The idea is that elements of an algebraically closed field (think complex numbers) are interpreted by pairs of real numbers.

Theorem (Alscher). $ACF_0 \not\prec RCF$.

Alscher's proof is based on Pillay's proof that a field K definable in an o-minimal structure is real closed iff $\dim K = 1$ and algebraically closed iff $\dim K = 2$, in "On Groups and Fields Definable in o-Minimal Structures" (1988).

The interpretation of elements of one theory by pairs of elements of another theory poses difficulties for the view that interpretations preserve meanings.

The results we just surveyed showed that points of Euclidean geometry, and complex numbers, are not interpreted by elements of RCF, but only by pairs of elements of RCF.

But pairs of elements of a structure are not themselves elements of the structure, strictly speaking.

If meaning is, at least in part, a matter of denotation, then the meaning has been changed by the interpretation.

Reply: the “structure” of Euclidean geometry or of the complex numbers is preserved by the interpretation: this is exactly what interpretability shows!

That’s to say, anything we do with points or complex numbers can be done with pairs in RCF, provability-wise.

If meaning is determined by inferential roles, as says the proof-theoretic semanticist, then interpretability in this case (and others) would indeed preserve meanings.

But as we will see, the case of pairing indicates that not everything provability-wise is preserved by interpretation.

Definition. A first-order theory T has pairing iff there is a 3-place predicate OP in the language of T (primitive or defined) and T proves

$$\forall xy \exists z(OP(x, y, z) \wedge \forall vw((OP(x, y, z) \wedge OP(v, w, z) \rightarrow x = v \wedge y = w))).$$

We can then investigate theories and their extensions by pairing, like RCF and RCF with pairing.

But RCF is a complete theory, while RCF with pairing is incomplete.

Proposition. If T is a first-order theory with pairing, with only infinite models, then T proves neither the following sentence expressing the surjectivity of pairing, nor its negation: $\forall z \exists xy OP(xyz)$.

So the reals and complexes thought of this way are not the same, provability-wise.

One might also hope to distinguish RCF and RCF with pairing by decidability.

As it turns out, this depends on how pairing is done.

For Cantor's pairing function $C(x, y) = \frac{(x+y)(x+y+1)}{2} + y$, Cegielski and Richard (1999) showed that $Th(\mathbb{N}, C)$ is undecidable.

Alscher (2015) showed that RCF with pairing defined by Cantor's pairing function is undecidable.

Since RCF is decidable, this is another way that RCF and RCF with pairing are different, provability-wise.

But not so fast.

For Ackemann's pairing function $A(x, y) = 2^{x+1} + 2^{x+y+2}$, Cegielski and Richard (1999) showed that $Th(\mathbb{N}, A)$ is decidable. This follows from:

Theorem (Semenov, 1983). The first-order theory of the structure of the natural numbers with addition and base-2 exponentiation is decidable.

So decidable theories extended by pairing do not always become undecidable.

We don't know any more general results on whether and, if so, for what classes of pairing functions, RCF with pairing is undecidable.

The following questions are a way to proceed.

Question. Are the integers definable in RCF with pairing?

Question. Is Robinson arithmetic interpretable in RCF with pairing?

We looked at two arguments that interpretations do not preserve meanings.

If they did, they would obliterate our usual understanding of the boundaries between mathematical domains, and as such render nonsensical the traditional search for purity.

Moreover, one canonical case of purity that raises interpretability questions, that of the complex numbers as ordered pairs of reals, also raises problems for understanding interpretations as meaning-preserving.

We thus counsel caution against doing so.