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The Active Role of Language Extensions in Mathematical Reasoning

Workshop on Proofs and Formalization in Logic, Mathematics and Philosophy

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Prologue

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One of my two aims in this talk is to present my own predicativist views and systems, which I believe are rather close to Weyl's original ones as reflected in Das Kontinuum.

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Adding a function symbol: Let φ be a formula in $\mathcal{L}$ such that $Fv(\varphi) = \{x_1, \ldots, x_n, y\}$, and let F be a new n-ary function symbol. Suppose that $\vdash_{\mathbf{T}} \forall x_1 \cdots \forall x_n \exists! y \varphi$. Then $\mathbf{T}$ can conservatively be extended to $\mathbf{T}^\star$ in $\mathcal{L} \cup \{F\}$ by the addition of either of the following axioms:

- $y = f(x_1, \ldots, x_n) \leftrightarrow \varphi$
- $\varphi\{f(x_1, \ldots, x_n)/y\}$

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- But it is well-known that the axiom of global choice implies AC, the usual axiom of choice of **ZFC**.
- It follows that $\vdash_{\mathbf{ZF}} AC$. Hence **ZF** and **ZFC** are equivalent!

Recursive Definitions — Two Approaches

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  2. The justification of transfinite recursion in standard textbooks on axiomatic set theories.

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In all these examples, the extension of S to the richer language is not conservative, but it is predicatively justified.

The Ideal Set Theory

Extensionality:

- $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$

The Comprehension Schema:

- $\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$

$\in$-induction:

- $(\forall x(\forall y(y \in x \rightarrow \varphi\{y/x\}) \rightarrow \varphi)) \rightarrow \forall x \varphi$

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Ideal, but inconsistent!

Giving up Some Ideals

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The Comprehension Schema:

- $\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$, if φ is safe w.r.t. x ($\varphi \succ \{x\}$).

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Principles of Poincaré-Weyl's Predicativism

- **Sets** (and functions) **are created only by definitions.**

"No one can describe an infinite set other than by indicating properties which are characteristic of the elements of the set. ... The notion that an infinite set as a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical;" [Weyl]

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- **Sets are "produced" genetically** [Weyl]. Therefore the elements of a set are "logically prior" to that set.

Principles of Poincaré-Weyl's Predicativism - Continued

- **A definition is predicative if the class it defines is invariant under extension.**

"Hence a distinction between two species of classifications, which are applicable to the elements of infinite collections: the predicative classifications, which cannot be disordered by the introduction of new elements; the non-predicative classifications, which are forced to remain without end by the introduction of new elements." [Poincaré]

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- Predicatively, the meaning is: "the identity of $\{\langle x_1, \dots, x_n\rangle \mid \varphi\}$ is stable: it depends only on the values assigned to $y_1, \dots, y_k$, but not on the surrounding universe.

Two Important Special Cases

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$n = 0$: φ is predicative with respect to $\emptyset$, if for every transitive S_1 and S_2 such that $S_1 \subseteq S_2$ and for every $b_1 \in S_1, \ldots, b_k \in S_1$:

$$S_2 \models \varphi(b_1, \ldots, b_k) \Leftrightarrow S_1 \models \varphi(b_1, \ldots, b_k)$$

I. e., if φ is absolute.

Basic (Set-theoretical) Conditions on Safety

- $\varphi \succ \emptyset$ if φ is atomic.
- $x{=}t \succ \{x\}$ if $x \notin Fv(t)$.
- $x \in t \succ \{x\}$ if $x \notin Fv(t)$ or t is x.
- $\neg\varphi \succ \emptyset$ if $\varphi \succ \emptyset$.
- $\varphi{\vee}\psi \succ X$ if $\varphi \succ X$ and $\psi \succ X$.
- $\varphi{\wedge}\psi \succ X \cup Y$ if $\varphi \succ X$, $\psi \succ Y$ and $Y \cap Fv(\varphi) = \emptyset$.
- $\exists y\varphi \succ X - \{y\}$ if $y \in X$ and $\varphi \succ X$.

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We denote by $\succ_{RST}$ the minimal relation which satisfies these conditions, and by *RST* (Rudimentary Set Theory) the set theory which is induced by $\succ_{RST}$.

The Power of *RST*

- $\emptyset =_{Df} \{x \mid x \in x\}$.
- $s - t =_{Df} \{x \mid x \in s \wedge \neg\, x \in t\}$
- $\{t_1, \ldots, t_n\} =_{Df} \{x \mid x = t_1 \vee \ldots \vee x = t_n\}$
- $\langle t, s\rangle =_{Df} \{\{t\}, \{t, s\}\}$.
- $\{x \in t \mid \varphi\} =_{Df} \{x \mid x \in t \wedge \varphi\}$, provided $\varphi \succ \emptyset$.
- $\{t(x) \mid x \in s\} =_{Df} \{y \mid \exists x. x \in s \wedge y = t\}$
- $\bigcup t =_{Df} \{x \mid \exists y. y \in t \wedge x \in y\}$
- $s \times t =_{Df} \{x \mid \exists a \exists b. a \in s \wedge b \in t \wedge x = \langle a, b\rangle\}$
- $\iota x \varphi =_{Df} \bigcup \{x \mid \varphi\}$ (provided $\varphi \succ \{x\}$).
- $\lambda x \in s.t =_{Df} \{\langle x, t\rangle \mid x \in s\}$
- $f(x) =_{Df} \iota y. \exists z \exists v (z \in f \wedge v \in z \wedge y \in v \wedge z = \langle x, y\rangle)$

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Full Replacement: $\exists y \varphi \wedge \forall y(\varphi \rightarrow \psi) \succ X$
provided $\psi \succ X$, and $X \cap Fv(\varphi) = \emptyset$.

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- In **RST** we have only that $N(x) \succ \emptyset$. In the new system, **RST**ω, we practically have $N(x) \succ \{x\}$. This means many more instances in the basic language of the schemas of **RST**.

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- Adding an n-ary function symbol F is allowed only if its defining axiom implies that if y is not free in $t_1, \ldots, t_n$ then the formula $y = F(t_1, \ldots, t_n)$ is safe with respect to $\{y\}$.
- As usual, extending **T** by a function symbol is allowed only if **T** proves some corresponding existence and uniqueness conditions. Still, the extension is usually not conservative.

Adding Predicate Symbols

Let $\mathcal{L}$ be a first-order language with equality which includes $\in$, let **T** be a theory in $\mathcal{L}$ which is based on the safety relation $\succ_{\mathcal{L}}$, and let $\mathcal{L}^*$ be the language which is obtained from $\mathcal{L}$ by the addition of a new $n+k$-ary predicate symbol P.

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A simple principle: Suppose that φ is a formula of $\mathcal{L}$ such that $Fv(\varphi) = \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$ and $\varphi \succ_{\mathcal{L}} \{y_1, \ldots, y_k\}$. Then we may extend $\mathbf{T}$ to a theory $\mathbf{T}^*$ in $\mathcal{L}^*$ by:

- Adding the axiom $P(x_1, \ldots, x_n, y_1, \ldots, y_k) \leftrightarrow \varphi$.
- Get $\succ_{\mathcal{L}^*}$ by adding to the definition of $\succ_{\mathcal{L}}$ the condition: $P(x_1, \ldots, x_n, y_1, \ldots, y_k) \succ_{\mathcal{L}^*} \{y_1, \ldots, y_k\}$
- Extending all the axiom schemas of $\mathbf{T}$ to $\mathcal{L}^*$, using $\succ_{\mathcal{L}^*}$ instead of $\succ_{\mathcal{L}}$.

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- Extending all the axiom schemas of $\mathbf{T}$ to $\mathcal{L}^*$, using $\succ_{\mathcal{L}^*}$ instead of $\succ_{\mathcal{L}}$.

An example: add $\subseteq$ to **RST** together with the axiom $x \subseteq y \leftrightarrow \neg\exists z(z \in x \wedge z \notin y)$ and the condition: $x \subseteq y \succ \emptyset$.

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1. φ has no subformula of the form $P(x_1 \ldots, x_n, v_1, \ldots, v_k)$;
2. Let $u \notin \{x_1 \ldots, x_n\}$ and let $1 \leq i, j \leq n$. If both $P(x_1, \ldots, x_{i-1}, u, \ldots)$ and $P(x_1, \ldots, x_{j-1}, u, \ldots)$ are subformulas of φ then $i = j$.
3. If $P(x_1, \ldots, x_{i-1}, u, \ldots)$ is a subformula of φ, where $u \notin \{x_1 \ldots, x_n\}$ and $i \geq 1$, then u is bound in φ by $\exists u \in x_i$.
4. $\varphi \succ_{\mathcal{L}^\star} \{y_1, \ldots, y_k\}$.

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4. $\varphi \succ_{\mathcal{L}^\star} \{y_1, \ldots, y_k\}$.

Under these conditions, P is uniquely defined, and is stable.

Examples

Transitive closure of $\in$: Add to the language the unary predicate symbol $\in^\star$, together with the condition $y \in^\star x \succ \{y\}$ and the axiom: $y \in^\star x \leftrightarrow y \in x \vee \exists z \in x. y \in^\star z$.

Examples

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The Graph of $+$: Add to the language the ternary predicate symbol P_+, with the condition $P_+(a,b,c) \succ \emptyset$ and the axiom:

$$\begin{aligned}&P_+(a,b,c) \leftrightarrow (b = \emptyset \wedge c = a) \vee \\ &\quad (a \in c \wedge \\ &\quad \forall x \in c (a \leq x \rightarrow \exists y \in b\ P_+(a,y,x)) \wedge \\ &\quad \forall z \in b \exists w \in c\ P_+(a,z,w))\end{aligned}$$

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This is not sufficient, though!

Adding Function Symbols

Let $\mathcal{L}$ be a first-order language with equality which includes $\in$, let $\mathbf{T}$ be a theory in $\mathcal{L}$ which is based on the safety relation $\succ_{\mathcal{L}}$, and let $\mathcal{L}^*$ be the language which is obtained from $\mathcal{L}$ by the addition of a new n-ary function symbol F.

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A weak principle: Suppose that φ is a formula of $\mathcal{L}$ such that $Fv(\varphi) = \{x_1, \ldots, x_n, y\}$ and $\varphi \succ_{\mathcal{L}} \{y\}$. If $\vdash_{\mathbf{T}} \forall x_1, \ldots, \forall x_n \exists! y \varphi$ Then we may extend $\mathbf{T}$ to a theory $\mathbf{T}^*$ in $\mathcal{L}^*$ by:

- Adding the axiom $\forall x_1, \ldots, \forall x_n \varphi\{F(x_1, \ldots, x_n / y\}$.
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Examples:

1. $\bigcup$, $\times$, ...
2. $\forall x[\forall z(z \in TC(x) \leftrightarrow z \in^{\star} x)]$

Adding Function Symbols — Continued

A much stronger principle is obtained from the weak one if instead of $\varphi \succ_{\mathcal{L}} \{y\}$ we demand only that $\varphi \succ_{\mathcal{L}} \emptyset$.

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Example: Any theory $\mathbf{T}$ which extends **RST** and includes P_+ proves uniqueness: $P_+(\alpha, \beta, \gamma_1) \wedge P_+(\alpha, \beta, \gamma_2) \rightarrow \gamma_1 = \gamma_2$. If it proves also existence: $\forall \alpha, \beta \exists \gamma P_+(\alpha, \beta, \gamma)$ then we can strengthen $\mathbf{T}$ by the addition of the corresponding function symbol $+$.

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For this $\in$-induction frequently does not suffice, though. We need a new, predicatively justified, principle.

Predicativity Beyond Γ_0

Feferman's Unification Rule: If $\varphi \succ \emptyset$, then

From:

$$\forall x \in a \forall y_1 \forall y_2 . \varphi\{y_1/y\} \wedge \varphi\{y_2/y\} \rightarrow y_1 = y_2$$

infer:

$$\forall x \in a \exists y \varphi \rightarrow \exists f (Function(f) \wedge Dom(f) = a \wedge \forall x \in a . \varphi(x, f(x)))$$

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Using this rule and our principles for language extensions, we can develop predicative set theories which have terms not only for Γ_0, but also for $\Gamma(\Gamma_0)$, and for much, much bigger ordinals.