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The Active Role of Language Extensions in Mathematical Reasoning

Workshop on Proofs and Formalization in Logic, Mathematics and Philosophy

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Prologue

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each anti-platonist is unhappy in her/his own way...

One of my two aims in this talk is to present my own predicativist views and systems, which I believe are rather close to Weyl’s original ones as reflected in Das Kontinuum.
Explicit Extensions by Definitions

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**Adding a function symbol:** Let $\varphi$ be a formula in $\mathcal{L}$ such that $Fv(\varphi) = \{x_1, \ldots, x_n, y\}$, and let $F$ be a new $n$-ary function symbol. Suppose that $\vdash \mathbf{T} \forall x_1 \cdots \forall x_n \exists! y \varphi$. Then $\mathbf{T}$ can conservatively be extended to $\mathbf{T}^*$ in $\mathcal{L} \cup \{F\}$ by the addition of either of the following axioms:

- $y = f(x_1, \ldots, x_n) \leftrightarrow \varphi$
- $\varphi\{f(x_1, \ldots, x_n)/y\}$
Let ϕ be a formula in L such that Fv(ϕ) = \{x_1, \ldots, x_n, y\}, and let
F be a new n-ary function symbol. Suppose that
\[ \vdash T \forall x_1 \cdots \forall x_n \exists y \varphi. \]
Then T can conservatively be extended to T^* in L \cup \{F\} by the addition of the axiom \[ \varphi\{f(x_1, \ldots, x_n)/y\}. \]

Puzzle:
Let $\varphi$ be a formula in $\mathcal{L}$ such that $Fv(\varphi) = \{x_1, \ldots, x_n, y\}$, and let $F$ be a new $n$-ary function symbol. Suppose that $\vdash_T \forall x_1 \cdots \forall x_n \exists y \varphi$. Then $T$ can conservatively be extended to $T^*$ in $\mathcal{L} \cup \{F\}$ by the addition of the axiom $\varphi\{f(x_1, \ldots, x_n)/y\}$.

**Puzzle:**

- $\vdash_{ZF} \forall x(x \neq \emptyset \rightarrow \exists y. y \in x)$
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- $\vdash_{ZF} \forall x(x \neq \emptyset \rightarrow \exists y.y \in x)$
- By Skolemization, we get that the axiom of global choice, $\forall x(x \neq \emptyset \rightarrow \epsilon(x) \in x)$, can conservatively be added to $ZF$. 

It follows that $\vdash_{ZF} AC$. Hence $ZF$ and $ZFC$ are equivalent!
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- But it is well-known that the axiom of global choice implies $AC$, the usual axiom of choice of $ZFC$. 
Skolemization

Let \( \varphi \) be a formula in \( \mathcal{L} \) such that \( \text{Fv}(\varphi) = \{x_1, \ldots, x_n, y\} \), and let \( F \) be a new \( n \)-ary function symbol. Suppose that \( \vdash_T \forall x_1 \cdots \forall x_n \exists y \varphi \). Then \( T \) can conservatively be extended to \( T^* \) in \( \mathcal{L} \cup \{F\} \) by the addition of the axiom \( \varphi\{f(x_1, \ldots, x_n)/y\} \).

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- It follows that \( \vdash_{ZF} AC \). Hence \( ZF \) and \( ZFC \) are equivalent!
Recursive Definitions — Two Approaches

1) Using induction to justify recursion: The use of recursive definitions of functions and predicates in a system $S$ is justified only if appropriate existence and uniqueness theorems are proved first in $S$ or in a stronger version $S^\star$ of it. Such proofs use principles of induction which are available in $S$ or $S^\star$, and are frequently impredicative. If $S$ is first-order then $S^\star$ is usually either the second-order version of it, or its meta-theory.

Examples:
1. The introduction of $+$ in the books of Landau and Mendelson on the foundations of Analysis.
2. The justification of transfinite recursion in standard textbooks on axiomatic set theories.
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Recursive Definitions — Two Approaches (Continued)

The use of recursive definitions of functions and predicates in a system $S$ is justified on the same ground that the use of the corresponding induction principle of $S$ is justified; no further justification is needed. Once the language of $S$ is extended by axioms of a recursive definition, the induction of $S$ is automatically extended to the expanded language.

Examples:
1. Primitive Recursive Arithmetic (PRA).
2. Weyl's iteration operation in "Das Kontinuum".
3. Adding truth-definition to PA and other systems. In all these examples, the extension of $S$ to the richer language is not conservative, but it is predicatively justified.
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The Ideal Set Theory

Extensionality:

\[ \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y \]

The Comprehension Schema:

\[ \forall x (x \in \{x \mid \varphi\} \leftrightarrow \varphi) \]

\[ \in\text{-induction:} \]

\[ (\forall x (\forall y (y \in x \rightarrow \varphi\{y/x\}) \rightarrow \varphi)) \rightarrow \forall x \varphi \]
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Ideal, but inconsistent!
Giving up Some Ideals

Extensionality:

\[ \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y \]

The Comprehension Schema:

\[ \forall x (x \in \{x \mid \varphi\} \leftrightarrow \varphi), \text{ if } \varphi \text{ is safe w.r.t. } x \]

\[ (\varphi \not
\in \{x\}). \]

\[ \epsilon\text{-induction:} \]

\[ (\forall x (\forall y (y \in x \rightarrow \varphi\{y/x\}) \rightarrow \varphi)) \rightarrow \forall x \varphi \]
Sets (and functions) are created only by definitions.

“No one can describe an infinite set other than by indicating properties which are characteristic of the elements of the set. . . . The notion that an infinite set as a “gathering” brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical;” [Weyl]
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- **Sets are “produced” genetically** [Weyl]. Therefore the elements of a set are “logically prior” to that set.
A definition is predicative if the class it defines is invariant under extension.

“Hence a distinction between two species of classifications, which are applicable to the elements of infinite collections: the predicative classifications, which cannot be disordered by the introduction of new elements; the non-predicative classifications, which are forced to remain without end by the introduction of new elements.”  

[Poincaré]
A set theory is determined by its safety relation $\succ$. 

The meaning of $\phi(x_1, \ldots, x_n, y_1, \ldots, y_k) \succ \{x_1, \ldots, x_n\}$ is:

"The collection $\{\langle x_1, \ldots, x_n \rangle \mid \phi\}$ is an acceptable set for all acceptable values of $y_1, \ldots, y_k$. Predicatively, the meaning is: "the identity of $\{\langle x_1, \ldots, x_n \rangle \mid \phi\}$ is stable: it depends only on the values assigned to $y_1, \ldots, y_k$, but not on the surrounding universe."
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Two Important Special Cases

\( k = 0: \) \( \varphi \) is predicative with respect to \( Fv(\varphi) \) iff it is domain independent in the sense of database theory.
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\[ n = 0: \] \( \varphi \) is predicative with respect to \( \emptyset \), if for every transitive \( S_1 \) and \( S_2 \) such that \( S_1 \subseteq S_2 \) and for every \( b_1 \in S_1, \ldots, b_k \in S_1 \):

\[ S_2 \models \varphi(b_1, \ldots, b_k) \iff S_1 \models \varphi(b_1, \ldots, b_k) \]

I. e., if \( \varphi \) is absolute.
Basic (Set-theoretical) Conditions on Safety

- $\varphi \succ \emptyset$ if $\varphi$ is atomic.
- $x=t \succ \{x\}$ if $x \not\in \text{Fv}(t)$.
- $x \in t \succ \{x\}$ if $x \not\in \text{Fv}(t)$ or $t$ is $x$.
- $\neg \varphi \succ \emptyset$ if $\varphi \succ \emptyset$.
- $\varphi \lor \psi \succ X$ if $\varphi \succ X$ and $\psi \succ X$.
- $\varphi \land \psi \succ X \cup Y$ if $\varphi \succ X$, $\psi \succ Y$ and $Y \cap \text{Fv}(\varphi) = \emptyset$.
- $\exists y \varphi \succ X - \{y\}$ if $y \in X$ and $\varphi \succ X$.
Basic (Set-theoretical) Conditions on Safety

- $\varphi \succ \emptyset$ if $\varphi$ is atomic.
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- $\varphi \land \psi \succ X \cup Y$ if $\varphi \succ X$, $\psi \succ Y$ and $Y \cap Fv(\varphi) = \emptyset$.
- $\exists y \varphi \succ X - \{y\}$ if $y \in X$ and $\varphi \succ X$.

We denote by $\succ_{RST}$ the minimal relation which satisfies these conditions, and by $RST$ (Rudimentary Set Theory) the set theory which is induced by $\succ_{RST}$.
The Power of RST

- \( \emptyset =_{Df} \{x \mid x \in x \} \).
- \( s - t =_{Df} \{x \mid x \in s \land \neg x \in t \} \).
- \( \{t_1, \ldots, t_n\} =_{Df} \{x \mid x = t_1 \lor \ldots \lor x = t_n \} \).
- \( \langle t, s \rangle =_{Df} \{\{t\}, \{t, s\}\} \).
- \( \{x \in t \mid \varphi\} =_{Df} \{x \mid x \in t \land \varphi\} \), provided \( \varphi \succ \emptyset \).
- \( \{t(x) \mid x \in s\} =_{Df} \{y \mid \exists x. x \in s \land y = t\} \).
- \( \bigcup t =_{Df} \{x \mid \exists y. y \in t \land x \in y\} \).
- \( s \times t =_{Df} \{x \mid \exists a \exists b. a \in s \land b \in t \land x = \langle a, b \rangle\} \).
- \( \iota x \varphi =_{Df} \bigcup \{x \mid \varphi\} \) (provided \( \varphi \succ \{x\}\)).
- \( \lambda x \in s. t =_{Df} \{\langle x, t \rangle \mid x \in s\} \).
- \( f(x) =_{Df} \iota y. \exists z \exists v(z \in f \land v \in z \land y \in v \land z = \langle x, y \rangle) \).
Handling Other Comprehension Axioms

Each of the other Comprehension axioms of ZF can be captured (in a modular way) by adding to the definition of $\succ_{RST}$ a certain syntactic condition:
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Handling Other Comprehension Axioms

Each of the other Comprehension axioms of ZF can be captured (in a modular way) by adding to the definition of $\succ_{RST}$ a certain syntactic condition:

Full Separation: $\varphi \succ \emptyset$ for every formula $\varphi$.

Powerset: $x \subseteq t \succ \{x\}$ if $x \notin Fv(t)$.

Full Replacement: $\exists y \varphi \land \forall y (\varphi \to \psi) \succ X$

provided $\psi \succ X$, and $X \cap Fv(\varphi) = \emptyset$. 
Let $S(x) = x \cup \{x\}$. 

The Axiom of Infinity: Introducing $\omega$
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- Define: $N(x) := \forall y \in S(x)(y = \emptyset \lor \exists z \in x.y = S(z))$
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- Obviously, the collection $[x:N(x)]$ is stable.
- This justifies the addition to the language of a new constant $\omega$, together with the following axiom:

$$\forall x(x \in \omega \leftrightarrow N(x))$$
The Axiom of Infinity: Introducing $\omega$

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- Obviously, the collection $[x:N(x)]$ is stable.
- This justifies the addition to the language of a new constant $\omega$, together with the following axiom:

\[ \forall x(x \in \omega \leftrightarrow N(x)) \]

- In $\text{RST}$ we have only that $N(x) \succ \emptyset$. In the new system, $\text{RST}\omega$, we practically have $N(x) \succ \{x\}$. This means many more instances in the basic language of the schemas of $\text{RST}$. 
Our Framework for Predicative Set Theories

Our main method of extending a given predicative set theory $T$ to a stronger one is by adding a new symbol to the signature of $T$, together with an axiom that defines it. Adding an $n$-ary predicate symbol $P$ is allowed only if its defining axiom implies its absoluteness. Like in the case of $t_1 = t_2$ and $t_1 \in t_2$, stronger safety conditions might hold for some atomic formulas of the form $P(t_1, \ldots, t_n)$.

Adding an $n$-ary function symbol $F$ is allowed only if its defining axiom implies that if $y$ is not free in $t_1, \ldots, t_n$ then the formula $y = F(t_1, \ldots, t_n)$ is safe with respect to $\{y\}$.

As usual, extending $T$ by a function symbol is allowed only if $T$ proves some corresponding existence and uniqueness conditions. Still, the extension is usually not conservative.
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- As usual, extending \( T \) by a function symbol is allowed only if \( T \) proves some corresponding existence and uniqueness conditions. Still, the extension is usually not conservative.
Let $\mathcal{L}$ be a first-order language with equality which includes $\in$, let $T$ be a theory in $\mathcal{L}$ which is based on the safety relation $\succ_{\mathcal{L}}$, and let $\mathcal{L}^*$ be the language which is obtained from $\mathcal{L}$ by the addition of a new $n + k$-ary predicate symbol $P$. 
Adding Predicate Symbols

Let \( L \) be a first-order language with equality which includes \( \in \), let \( T \) be a theory in \( L \) which is based on the safety relation \( \succ_L \), and let \( L^* \) be the language which is obtained from \( L \) by the addition of a new \( n + k \)-ary predicate symbol \( P \).

**A simple principle:** Suppose that \( \varphi \) is a formula of \( L \) such that \( Fv(\varphi) = \{x_1, \ldots, x_n, y_1, \ldots, y_k\} \) and \( \varphi \succ_L \{y_1, \ldots, y_k\} \). Then we may extend \( T \) to a theory \( T^* \) in \( L^* \) by:

- Adding the axiom \( P(x_1, \ldots, x_n, y_1, \ldots, y_k) \leftrightarrow \varphi \).
- Get \( \succ_{L^*} \) by adding to the definition of \( \succ_L \) the condition:
  \[
P(x_1, \ldots, x_n, y_1, \ldots, y_k) \succ_{L^*} \{y_1, \ldots, y_k\}
  \]
- Extending all the axiom schemas of \( T \) to \( L^* \), using \( \succ_{L^*} \) instead of \( \succ_L \).
Adding Predicate Symbols

Let $\mathcal{L}$ be a first-order language with equality which includes $\in$, let $T$ be a theory in $\mathcal{L}$ which is based on the safety relation $\succ_{\mathcal{L}}$, and let $\mathcal{L}^*$ be the language which is obtained from $\mathcal{L}$ by the addition of a new $n + k$-ary predicate symbol $P$.

A simple principle: Suppose that $\varphi$ is a formula of $\mathcal{L}$ such that $\text{Fv}(\varphi) = \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$ and $\varphi \succ_{\mathcal{L}} \{y_1, \ldots, y_k\}$. Then we may extend $T$ to a theory $T^*$ in $\mathcal{L}^*$ by:

- Adding the axiom $P(x_1, \ldots, x_n, y_1, \ldots, y_k) \iff \varphi$.
- Get $\succ_{\mathcal{L}^*}$ by adding to the definition of $\succ_{\mathcal{L}}$ the condition: $P(x_1, \ldots, x_n, y_1, \ldots, y_k) \succ_{\mathcal{L}^*} \{y_1, \ldots, y_k\}$
- Extending all the axiom schemas of $T$ to $\mathcal{L}^*$, using $\succ_{\mathcal{L}^*}$ instead of $\succ_{\mathcal{L}}$.

An example: add $\subseteq$ to RST together with the axiom $x \subseteq y \iff \neg \exists z (z \in x \land z \notin y)$ and the condition: $x \subseteq y \succ \emptyset$. 
A stronger principle: We can similarly extend $T$ to $T^*$ as above even if $\varphi$ is a formula of $L^*$, provided that:

1. $\varphi$ has no subformula of the form $P(x_1, \ldots, x_n, v_1, \ldots, v_k)$;
2. Let $u \not\in \{x_1, \ldots, x_n\}$ and let $1 \leq i, j \leq n$. If both $P(x_1, \ldots, x_{i-1}, u, \ldots)$ and $P(x_1, \ldots, x_{j-1}, u, \ldots)$ are subformulas of $\varphi$ then $i = j$.
3. If $P(x_1, \ldots, x_{i-1}, u, \ldots)$ is a subformula of $\varphi$, where $u \not\in \{x_1, \ldots, x_n\}$ and $i \geq 1$, then $u$ is bound in $\varphi$ by $\exists u \in x_i$.
4. $\varphi \succ L^* \{y_1, \ldots, y_k\}$.

Under these conditions, $P$ is uniquely defined, and is stable.
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Under these conditions, $P$ is uniquely defined, and is stable.
Transitive closure of $\in$: Add to the language the unary predicate symbol $\in^*$, together with the condition $y \in^* x \supset \{y\}$ and the axiom: $y \in^* x \iff y \in x \lor \exists z \in x. y \in^* z.$
Examples

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The Graph of $\vdash$: Add to the language the ternary predicate symbol $P_\vdash$, with the condition $P_\vdash(a, b, c) \supset \emptyset$ and the axiom:

$$P_\vdash(a, b, c) \iff (b = \emptyset \land c = a) \lor (a \in c \land \forall x \in c(a \leq x \rightarrow \exists y \in b P_\vdash(a, y, x)) \land \forall z \in b \exists w \in c P_\vdash(a, z, w))$$

Similarly, we can define the relations $P_{\lambda x.\omega^x}$, $P_{\phi}$, and $P_{\Gamma}$ as absolute relations (on ordinals).
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\[
P_+(a, b, c) \iff (b = \emptyset \land c = a) \lor \\
(a \in c \land \\
\forall x \in c (a \leq x \rightarrow \exists y \in b P_+(a, y, x)) \land \\
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Similarly, we can define the relations $P_{\lambda x.\omega^x}$, $P_\phi$, and $P_\Gamma$ as absolute relations (on ordinals).

This is not sufficient, though!
Adding Function Symbols

Let $\mathcal{L}$ be a first-order language with equality which includes $\in$, let $\mathbf{T}$ be a theory in $\mathcal{L}$ which is based on the safety relation $\succ_{\mathcal{L}}$, and let $\mathcal{L}^*$ be the language which is obtained from $\mathcal{L}$ by the addition of a new $n$-ary function symbol $F$. 

A weak principle: Suppose that $\varphi$ is a formula of $\mathcal{L}$ such that $\text{Fv}(\varphi) = \{x_1,\ldots,x_n,y\}$ and $\varphi \succ_{\mathcal{L}} \{y\}$. If $\vdash_{\mathbf{T}} \forall x_1,\ldots,\forall x_n \exists ! y \varphi$ Then we may extend $\mathbf{T}$ to a theory $\mathbf{T}^*$ in $\mathcal{L}^*$ by:

- Adding the axiom $\forall x_1,\ldots,\forall x_n \varphi\{F(x_1,\ldots,x_n/y)\}$.
- Extending all the axiom schemas of $\mathbf{T}$ to $\mathcal{L}^*$, using $\succ_{\mathcal{L}^*}$ instead of $\succ_{\mathcal{L}}$. 

Examples:

1. $\bigcup$, $\times$, ... 
2. $\forall x \left[ \forall z (z \in \text{TC}(x) \iff z \in \star x) \right]$
Adding Function Symbols

Let $\mathcal{L}$ be a first-order language with equality which includes $\in$, let $\mathbf{T}$ be a theory in $\mathcal{L}$ which is based on the safety relation $\triangleright_{\mathcal{L}}$, and let $\mathcal{L}^*$ be the language which is obtained from $\mathcal{L}$ by the addition of a new $n$-ary function symbol $F$.

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**Examples:**

1. $\bigcup, \times, \ldots$
2. $\forall x[\forall z(z \in \mathcal{T}C(x) \leftrightarrow z \in^* x)]$
A much stronger principle is obtained from the weak one if instead of $\varphi \succ_{\mathcal{L}} \{y\}$ we demand only that $\varphi \succ_{\mathcal{L}} \emptyset$. 
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The resulting $\mathbf{T}^*$ is in this case not a conservative extension of $\mathbf{T}$, and it allows to define many new sets. Nevertheless, it is still predicatively justified by the stability criterion.
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Example: Any theory $T$ which extends $\text{RST}$ and includes $P_+$ proves uniqueness: $P_+(\alpha, \beta, \gamma_1) \land P_+(\alpha, \beta, \gamma_2) \rightarrow \gamma_1 = \gamma_2$. If it proves also existence: $\forall \alpha, \beta \exists \gamma P_+(\alpha, \beta, \gamma)$ then we can strengthen $T$ by the addition of the corresponding function symbol $+$. 
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For this $\varepsilon$-induction frequently does not suffice, though. We need a new, predicatively justified, principle.
Feferman’s Unification Rule: If $\varphi \succ \emptyset$, then

From:

$$\forall x \in a \forall y_1 \forall y_2. \varphi\{y_1/y\} \wedge \varphi\{y_2/y\} \rightarrow y_1 = y_2$$

infer:

$$\forall x \in a \exists y \varphi \rightarrow \exists f (Function(f) \wedge Dom(f) = a \wedge \forall x \in a. \varphi(x, f(x)))$$

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Using this rule and our principles for language extensions, we can develop predicative set theories which have terms not only for $\Gamma_0$, but also for $\Gamma(\Gamma_0)$, and for much, much bigger ordinals.