On the semantics for relevance logic

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Workshop on Proofs and Formalization in Logic, Mathematics and Philosophy, Utrecht
September 20–22, 2022

Relevance and contradictions

Logical consequence & implication

According to the modern account:

NTP: An inference is good iff it necessarily preserves truth. I.e. a necessary and sufficient condition for inferential goodness is necessary truth preservation.

Classically, the following are therefore valid:

$$\text{DS}\ \frac{A \qquad \neg A \vee B}{B} \qquad\qquad \text{EFQ}\ \frac{A \wedge \neg A}{B}$$

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Likewise, according to the modern classical account:

TP: An implication $A \to B$ is true iff it preserves truth.

The following are therefore valid:

- $A \to (B \to A)$
- $A \to (B \vee \neg B)$

C. I. Lewis on relevance

True postulates may materially [but not logically] imply theorems which are quite irrelevant [. . .] "Proof" requires that a connection of content or meaning or logical significance be established [. . .] For a relation which does not indicate relevance of content is merely a connection of "truth values", not what we mean by a "logical" relation or "inference". (Lewis 1917)

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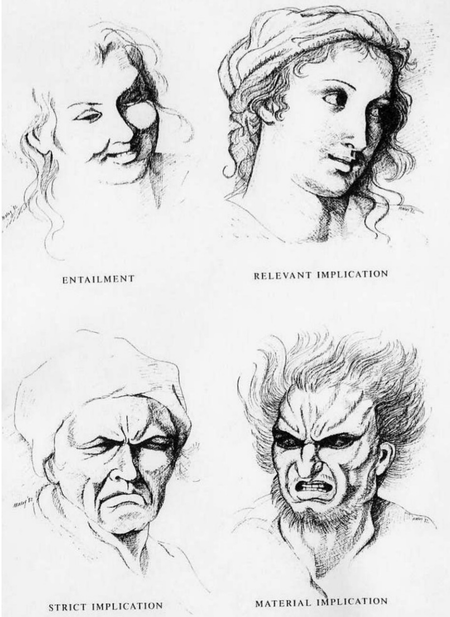
So Lewis held TP but seems to reject the sufficiently half of NTP.

However, it seems he gave up on relevance and instead settled for something halfway between material and relevant implication.

Four types of implication



Anderson & Belnap on relevance

For more than two millennia logicians have taught that a necessary condition for the validity of an inference from A to B is that A be relevant to B. Virtually every logic book up to the present century has a chapter on fallacies of relevance, and many contemporary elementary texts have followed the same plan. Notice that contemporary writers, in the later and more formal chapters of their books, seem explicitly to contradict the earlier chapters, when they try desperately to bamboozle the students into accepting strict "implication" as a "kind" of implication relation, in spite of the fact that this relation countenances fallacies of relevance. (Anderson & Belnap 1975)

DS and EFQ: fallacies of relevance

The principle 4 (from $\neg A \vee B$ and A to infer B) [...] **commits a fallacy of relevance**. *We therefore reject 4 as an entailment and as a valid principle of inference [...] [W]e agree at once that the inference from A and $\neg A \vee B$ to B is valid in [the sense that it necessarily preserves truth]. (Anderson & Belnap 1975)*

[W]e should have agreed at once that there is a valid form of inference from $A \wedge \neg A$ to B: it is surely true that necessarily either the premiss is false or the conclusion is true inasmuch as the premiss is necessarily false. (Anderson & Belnap 1975)

Fallacies of relevance, not fallacies of NTP

According to traditional relevantists, inferences such as EFQ and DS:

1. necessarily preserve truth, but
2. commit a fallacy of relevance

A classically valid inference is invalid just in case it commits a fallacy of relevance.

Natural deduction for relevant implication

The first natural deduction system for relevance logic provided a neat separation between truth preservation and relevance. In order to mark the **use and hence relevance** of a hypothesis in a derivation, subscript it with a numeral. Then the rules for the conditional require that in concluding $A \to B$, A must have been used in deriving B.

$$(\to\mathrm{E})\ \frac{A_\alpha \qquad A \to B_\beta}{B_{\alpha\cup\beta}}$$

$$\begin{array}{c}[A_{\{k\}}]\\ \vdots \\ (\to\mathrm{I})\ \dfrac{B_\alpha}{A \to B_{\alpha-\{k\}}}\end{array}$$

where for $(\to\mathrm{I})$, k must occur in α.

Examples

Example proof:

1.	A_1	hypothesis
2.	$A \to B_2$	hypothesis
3.	$B_{\{1,2\}}$	1,2 $\to$E
4.	$(A \to B) \to B_1$	2–3 $\to$I
5.	$A \to ((A \to B) \to B)$	1–4 $\to$I

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Example of a failed proof:

1.	A_1	hypothesis
2.	B_2	hypothesis
3.	$B \to A_1$	1–2 $\to$I
4.	$A \to (B \to A)$	1–3 $\to$I

Syntax vs "set-theoretical garbage"

Once upon a time, modal logics "had no semantics". Bearing a real world G, a set of worlds K, and a relation R of relative possibility between worlds, Saul Kripke beheld this situation and saw that it was formally explicable, and made model structures. It came to pass that soon everyone was making model structures, and some were deontic, and some were temporal, and some were epistemic, according to the conditions on the binary relation R. None of the model structures that Kripke made, nor that Hintikka made, nor that Thomason made, nor that their co-workers and colleagues made, were, however, relevant. This caused great sadness in the city of Pittsburgh, where dwelt the captains of American Industry. The logic industry was there represented by Anderson, Belnap & Sons, discoverers of entailment and scourge of material impliers, strict impliers, and of all that to which their falsehoods and contradictions led. Yea, every year or so Anderson & Belnap turned out a new logic [...] and they beheld each such logic, and they were called relevant. And these logics were looked upon with favor by many, for they captureth the intuitions, but by many more they were scorned, in that they hadeth no semantics. Word that Anderson & Belnap had made a logic without semantics leaked out. Some thought it wondrous and rejoiced, that the One True Logic should make its appearance among us in the Form of Pure Syntax, unencumbered by all that set-theoretical garbage. Others said that relevant logics were Mere Syntax. (Routley & Meyer 1973)

Routley & Meyer on the semantics for relevance logic

Negation, on the other hand, requires as in Routley 1972, as in previous work of Dunn, Belnap, and others, the admission of theories that are inconsistent, incomplete, or both [...] the strategy which dispatches the paradoxes lies in allowing even logical identities to turn out sometimes false. (What, after all, could be better grounds for denying that q entails $p \to p$ than to admit that sometimes q is true when, essentially on grounds of relevance, $p \to p$ isn't?) (Routley & Meyer 1973)

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What does the last sentence mean? How can $p \to p$ fail to be true **on grounds of relevance**? If he means that it fails on grounds of relevance because $p \to p$ is irrelevant to q, then the "argument" loses all force.

Copeland's objection

The obvious way, then, to obtain inconsistent and incomplete structures (set ups) is to modify the usual treatment of negation [...] However, it is crucial that the semantics preserve the classical meaning of $\neg$, and indeed of all the connectives of classical propositional logic. [The requirement of relevance] will lead on to assign a stronger-than-classical meaning to $\rightarrow$, but the interest does not license, and indeed prohibits, a re-interpretation of any connectives of the language with whose inference patterns one is concerned [...] Routley and Meyer have a responsibility, then, to demonstrate that the innovatory treatment given to $\neg$ in their semantics nonetheless preserves its classical meaning. (Copeland 1979)

Relevant semantics for negation

There are two types of semantics for negation in relevance logic:

Australian Plan: $w \models \neg A$ iff $w* \not\models A$

American Plan: $w \models \neg A$ iff $w \models^{-} A$; $w \models^{-} \neg A$ iff $w \models A$

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A common definition of classical negation within a possible worlds semantics is:

Classical negation: $w \models \neg A$ iff $w \not\models A$

Routley's defense

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Copeland remains unsatisfied:

> *For doesn't it follow that $\neg$ is non-classical simply from the fact that $A \vee B$ and $\neg A$ can both be true in a model whilst B is false in that model? Indeed, what better evidence could there be for the non-classical nature of $\neg$ than that both A and $\neg A$ can be true in the same model. (Copeland 1979)*

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Copeland argues that relevant semantics should preserve the *meaning* of the classical connectives, including $\neg$, while assigning a different meaning *only to the conditional*. However, it assigns an *intensional* meaning to $\neg$.

Copeland continued

Thus the 'semantics' yields no semantical argument for the fallaciousness of the entailment (DS), for such an argument must show how the invalidity of (DS) arises from a classical account of the meanings of the propositional connectives together with a relevance account of the meaning of entailment. But, as we have seen, it is totally unclear what account of the meanings of the logical constants is given in the Routley-Meyer 'semantics'. (Copeland 1979)

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Copeland goes on to object to various interpretations of the $*$ used in the truth conditions given to $\neg$. E.g. he objects to Routley's explanation that $a*$ is the reverse of world a, where "a reverse world is like the reverse side of something, e.g. a gramophone record".

Dunn on the Routley star

Routleys perform some magic with a 'star operation' in giving the truth condition for negation. By a feat of prestidigitation one 'set up' H is switched with another set up H. Thus ¬A is true in H iff A is not true in H* (instead of the usual plain H). But just what is this 'star operation' and why does it stick its nose into the truth condition for negation? This seems to me to remain an ultimate mystery in the Routleys' semantics, and I count it as a philosophical virtue of my semantics that it does without the 'star operation'. (Dunn 1976)*

Assessing Copeland's objection

Should a treatment of the relevant conditional preserve the classical meaning of negation?

- Yes

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Should a treatment of the relevant conditional preserve the classical meaning of negation?

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Does a treatment of the relevant conditional preserve the classical meaning of negation?

- (Arguably) No

Slater's objection

Consider negation on the American plan:

- A is true, i.e. A, iff $v(A) = t$ or $v(A) = b$
- A is false, i.e. $\neg A$, iff $v(A) = f$ or $v(A) = b$

Clearly A can be both true and false, hence A and $\neg A$ can be both true. But then they are not contraries, so they are not contradictories, so $\neg$ is not a negation, i.e. a contradictory-forming operator.

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This objection is not as serious as Copeland's since one can easily take issue with the definition of *contradictoriness*. Still, it would be best if the relevantist could avoid it.

Say that A and B are contradictories iff:

Contrariety: If A is true then B is false

Subcontrariety: If A is false then B is true

Lewis on relevance

Lately, however, relevance has been praised not—or not only—as a separate merit, but rather as something needed to ensure preservation of truth. The trouble with fallacies of relevance, it turns out, is that they can take us from truth to error [. . .] Classical implication does preserve truth, to be sure, so long as sentences divide neatly into those that are true and those with true negations. But when the going gets tough, and we encounter true sentences whose negations also are true, then [. . .] relevant implication preserves truth and some classical implication doesn't. (Lewis 1982)

Lewis's objection

No truth does have, and no truth could have, a true negation. Nothing is, and nothing could be, literally both true and false. This we know for certain, and a priori, and without any exception for especially perplexing subject matters. The radical case for relevance should be dismissed just because the hypothesis it requires us to entertain is inconsistent. (Lewis 1982)

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Lewis thinks that a "vindication of relevance" requires finding a way "in which sentences can be regarded as true and false". But why? Can't one can naturally explain why $p \wedge \neg p$ is not relevant to q without providing an interpretation according to which $p \wedge \neg p$ is true?

What is logical implication?

Logical implication, one might think, concerns only “realizable situations”, not belief states or states of information or the like. But then if contradictions are not realizable, set ups that make contradictions true or fail to make tautologies true play no role in a semantics for logical implication.

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> *Intuitively, $p \wedge \neg p$ and $q \wedge \neg q$ describe different situations, granted that neither situation is realizable [...] Do not get me wrong—I am not claiming that there are sentences which are in fact both true and false. I am merely pointing out that there are plenty of situations where we suppose, assert, believe, etc., contradictory sentences to be true, and we therefore need a semantics which expresses the truth conditions of contradictions in terms of the truth values that the ingredient sentences would have to take for the contradictions to be true. (Dunn 1976)*

Relevance without true contradictions?

Semantics for content sharing (relevance)

Consider the 0-degree fragment of the language, i.e. sentences built up exclusively from the connectives $\wedge$, $\vee$, and $\neg$.

A content pre-assignment $c^* : \mathit{Atoms} \to \wp(X)$ assigns contents to atoms. A content assignment $c : \mathit{Sentences} \to \wp(X)$ assigns contents to sentences as follows:

- $c(p) = c^*(p)$ for atoms p
- $c(A \wedge B) = c(A) \cap c(B)$
- $c(A \vee B) = c(A) \cup C(B)$
- $c(\neg A) = c(A)$

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Define a pre-relevance relation $R^*_c(A, B)$ iff $c(A') \subseteq c(B')$, for all A' and B' that are DNFs of A and of B. Let R_c be the symmetric closure of R^*_c. Say that A and B are relevantly related, $R(A, B)$, iff for all c, $R_c(A, B)$. Finally, say that $A \to B$ holds if, (i) $A \models B$, and (ii) $R(A, B)$.

Some results

The following hold:

1. From $A \to B$ and $B \to C$ infer $A \to C$
2. $A \wedge B \to A$; $A \wedge B \to B$
3. From $A \to B$ and $A \to C$ infer $A \to B \wedge C$
4. $A \to A \vee B$; $B \to A \vee B$
5. From $A \to C$ and $B \to C$ infer $A \vee B \to C$
6. $A \wedge (B \vee C) \to (A \wedge B) \vee C$
7. $A \to \neg\neg A$; $\neg\neg A \to A$
8. $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$; $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
9. From $A \to B$ infer $\neg B \to \neg A$

The following fail:

1. $A \wedge (\neg A \vee B) \to B$
2. $A \wedge \neg A \to B$

Some results

- If $A \to B$ holds then A and B share a variable
- If $A \to B$ is derivable in $\mathbf{E}_{fde}$ then $A \to B$ holds
- I conjecture the converse is also true

For the future...

- Assigning contents to the full language (nesting of implications of arbitrary degree)
- Giving content algebras for various logics (e.g. **R**)
- Discovering new and interesting logics based on independently motivated content algebras
- Determining limitations of strategy