HYPE and Cuts

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Aims

- proof systems for the logic of HYPE
- sequent calculi for the propositional part
- suitable properties
- allow for cut-elimination

Plan for the talk

1. Preliminaries
   - HYPE
2. Sequent systems for HYPE
   - A G1 system
   - A G3 system
   - Problems for cut-elimination
   - A solution strategy by Kashima and Shimura
3. The case of HYPE
   - Reformulating HYPE
   - Cut-elimination
   - Equivalence

HYPE

- [Leitgeb 2019] introduced the logic for hyperintensional contexts
- application truth
- Hilbert style calculus
- FDE + intuitionistic conditional

Nice semantics allowing for gaps and gluts.
Routley-style semantics with an involutive $*$ function for $\neg$.
Soundness and Completeness.

The axioms of HYPE

Based on $\neg, \vee, \rightarrow, \bot$. The intuitionistic axioms:

$$
\begin{aligned}
&A \rightarrow (B \rightarrow A) && (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) && (1)\\
&A \wedge B \rightarrow A && A \wedge B \rightarrow B && (2)\\
&A \rightarrow A \vee B && B \rightarrow A \vee B && (3)\\
&A \rightarrow (B \rightarrow A \wedge B) && (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C)) && (4)\\
&\bot \rightarrow A && && (5)
\end{aligned}
$$

and the axioms for double negation:

$$A \rightarrow \neg\neg A \qquad\qquad \neg\neg A \rightarrow A \tag{6}$$

Closure under modus ponens and the rule of conditional contraposition:

$$\frac{\vdash A \rightarrow B}{\vdash \neg B \rightarrow \neg A}$$

(This formulation is due to [Speranski, 2021])

An application: Kripkean Truth

- Axiomatization: classical KF versus nonclassical PKF
- PKF over FDE is significantly weaker than KF

$$PKF \vdash TI_{\mathcal{L}_T}(< \omega^\omega)$$

$$KF \vdash TI_{\mathcal{L}_T}(< \varepsilon_0)$$

- a suitable conditional is missing to carry out Gentzen's proof
- the conditional of HYPE is suitable (introduction and elimination)
- PKF over HYPE is proof-theoretically equivalent to KF [F., Nicolai, Dopico 2021]
- Remark: because of the Curry-paradox the truth theoretic axioms are restricted to the conditional free-fragment.

Sequent systems for HYPE

A G1 system

$(ID_p) \qquad A \Rightarrow A \qquad\qquad (L\bot) \qquad \bot \Rightarrow$

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(LW)\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad (RW)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$(LC)\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad (RC)\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

$$(\mathrm{L}\vee)\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

The conditional

$$(\mathsf{L}\!\to)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad (\mathsf{R}\!\to)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

$$(\mathsf{ConCp})\ \frac{\Gamma \Rightarrow \neg\Delta}{\Delta \Rightarrow \neg\Gamma} \qquad (\mathsf{ClCp})\ \frac{\neg\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \Gamma}$$

The resulting system is labelled $\mathbf{G1h_p}$.

Equivalence

$\mathbf{G1h_p}$ and the axiomatic system of HYPE are equivalent.

$$\mathbf{G1h_p} \vdash \Gamma \Rightarrow \Delta \text{ iff } \mathit{HYPE} \vdash \bigwedge \Gamma \to \bigvee \Delta.$$

Remark: Simple, but the rules (ConCp) and (ClCp) are not suitable for a direct cut-elimination argument.
$\Rightarrow$ Admissible?

$\mathbf{G3h_p}$

For v a literal:

$$(ID) \qquad v, \Gamma \Rightarrow \Delta, v$$

$$(L\bot) \qquad \bot, \Gamma \Rightarrow \Delta \qquad\qquad (R\neg\bot) \qquad \Gamma \Rightarrow \Delta, \neg\bot$$

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\mathrm{L}\neg\neg)\ \frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta} \qquad\qquad (\mathrm{R}\ \neg\neg)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

$$(\mathrm{L}\vee)\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad\qquad (\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$(\mathrm{L}\to)\ \frac{A \to B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad (\mathrm{R}\to)\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B}$$

Negated rules

$$(\mathsf{L}\neg\vee)\ \frac{\neg A, \neg B, \Gamma \Rightarrow \Delta}{\neg(A \vee B), \Gamma \Rightarrow \Delta}$$

$$(\mathsf{R}\neg\vee)\ \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}$$

$$(\mathsf{L}\neg\rightarrow)\ \frac{\neg B \Rightarrow \Delta, \neg A}{\neg(A \rightarrow B), \Gamma \Rightarrow \Delta}$$

$$(R\neg\rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg B \qquad \neg A, \Gamma \Rightarrow \Delta, \neg(A \rightarrow B)}{\Gamma \Rightarrow \Delta, \neg(A \rightarrow B)}$$

Properties of $\mathbf{G3h_p}$

- All the rule of HYPE are symmetric and the propositional rules guarantee that the principal formula is of greater logical complexity as the active formulas.
- Weakening and contraction are admissible.

Lemma 1 (Admissibility of contraposition)

If $\mathbf{G3h_p} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathbf{G3h_p} \vdash \neg\Delta \Rightarrow \neg\Gamma$.

Equivalence

$\mathbf{G3h_p}$ and $\mathbf{G1h_p}$ are equivalent, i.e.

$$\mathbf{G3h_p} \vdash \Gamma \Rightarrow \Delta \text{ iff } \mathbf{G1h_p} \vdash \Gamma \Rightarrow \Delta.$$

Inversion

Observation

$(L\vee), (R\vee), (L\neg\vee), (R\neg\vee), (L\neg\neg), (R\neg\neg), (L \to), (R\neg \to)$ are invertible.

Remark: The rules $(R \to)$ and $(L\neg \to)$ are not invertible in contrast to the single conclusion calculus **G3i**
Problematic case:

$$\frac{\neg D \Rightarrow \neg C, A \to B}{\neg(C \to D) \Rightarrow A \to B}$$

It is not guaranteed that $\neg(C \to D), A \Rightarrow B$ is derivable.

Cut?

Is cut admissible?

- cut-elimination for the FDE part is straightforward;
- all the rules of $\mathbf{G3h_p}$ are symmetric;
- the logical complexity of principal formulas is greater than the complexity of the active formulas;
- contraposition is admissible.

Counterexample to cut-elimination

$$\neg(R \to \neg(P \to Q)), P \Rightarrow Q \tag{$\dagger$}$$

- there is no derivation of ($\dagger$) in $\mathbf{G3h_p}$ without cut (proof search).
- An application of $(L\neg\to)$ would require that there is no formula in the context, but $\neg\neg(P \to Q) \Rightarrow Q, \neg R$ is not derivable.

On the other hand the following are derivable:

$$\neg(R \to \neg(P \to Q)), P \Rightarrow P \to Q \tag{7}$$

$$P \to Q, \neg(R \to \neg(P \to Q)), P \Rightarrow Q \tag{8}$$

An application of cut to (7) and (8) gives ($\dagger$).

Some observations

$$\neg\neg(P \to Q), P \Rightarrow Q, \neg R \qquad (*)$$

- $(*)$ is cut-free derivable.
- Although $(\dagger)$ is not cut-free derivable in $\mathbf{G3h_p}$ it would be, if we could use a left negated conditional introduction on $(*)$;
- The occurrence of $\neg R$ in the succedent is introduced by weakening;
- $\neg R$ is independent of the occurrence of P in the antecedent.

Constant domains CD

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

$$\frac{\Gamma \Rightarrow \Delta, A(b)}{\Gamma \Rightarrow \Delta, \forall x A_b(x)} \text{ for } b \notin FV(\Gamma, \Delta, \forall x A_b(x))$$

The sequent

$$\forall x(B \vee A(x)) \Rightarrow B, \top \to \forall x A(x) \tag{‡}$$

is derivable with cut, but not without cut.
[López-Escobar 1983]

The solution by Kashima and Shimura

Instead of

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

use the more general

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \text{ if } A \text{ and } \Delta \text{ are independent (not connected)}$$

$\Rightarrow$ Introduce connections and keep track of the connections within derivations!

The case of HYPE

Connections

The occurrences of A in an initial sequent are connected:

$$A \Rightarrow A$$

Formulas introduced by weakening have no connection to other formulas.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

Keep track of the connections with a suitable labelling of the formula occurrences within derivations.

$$A^i[k_1, ..., k_n]$$

i is the label and
$[k_1, ..., k_n]$ are the labels of formula occurrences to which A^i is connected within a sequent.

Initial sequents

For all formulas A:

$$\begin{array}{rl} (ID) & A^i[j] \Rightarrow A^j[i] \\ (L\bot') & \bot^i[j] \Rightarrow A^j[i] \\ (R\neg\bot') & A^i[j] \Rightarrow (\neg\bot)^j[i] \end{array}$$

Remark: The additional formula A in $(L\bot')$ and $(R\neg\bot')$ is due to technical reasons (Lemma 4) avoiding empty succedents and antecedents.

Structural rules

$$(\mathrm{LW})\ \frac{\Gamma \Rightarrow \Delta}{A^i, \Gamma \Rightarrow \Delta}$$

$$(\mathrm{LC})\ \frac{A^i[1_i, \ldots, n_i], A^j[1_j, \ldots, m_j], \Gamma \Rightarrow \Delta}{A^k[1'_i, \ldots, n'_i, 1'_j, \ldots, m'_j], \Gamma \Rightarrow \Delta}$$

The $'$ indicates that the formula occurrence in the upper sequent with label k is a direct ancestor of the formula occurrence with label k' in the lower sequent.

Propositional rules

$$(\mathrm{L}\neg\neg)\ \frac{A^i[1_i, ..., n_i], \Gamma \Rightarrow \Delta}{(\neg\neg A)^i[1'_i, ..., n'_i], \Gamma \Rightarrow \Delta}$$

$$(\mathrm{L}\vee)\ \frac{A^i[1_i, ..., n_i], \Gamma \Rightarrow \Delta \qquad B^j[1_j, ..., m_j], \Pi \Rightarrow \Lambda}{(A \vee B)^k[1'_i, ..., n'_i, 1'_j, ..., m'_j], \Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

$$(\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow \Delta, A^i[1_i, ..., n_i]}{\Gamma \Rightarrow \Delta, (A \vee B)^i[1'_i, ..., n'_i]}$$

generalized rules for the conditional

$$(R \rightarrow^+) \; \frac{A^i[1_i, ..., n_i], \Gamma \Rightarrow \Delta, B^j[1_j, ..., m_j]}{C^l[k], \Gamma \Rightarrow \Delta, (A \rightarrow B)^k[l, 1'_j, ..., m'_j \setminus i]} \text{ for all } \delta \in \Delta,\ \delta \notin []^i$$

$$(\mathsf{L}\neg\rightarrow^+) \; \frac{(\neg B)^i[1_i, ..., n_i], \Gamma \Rightarrow \Delta, (\neg A)^j[1_j, ..., m_j]}{(\neg(A \rightarrow B))^k[l, 1'_i, ..., n'_i \setminus j], \Gamma \Rightarrow \Delta, C^l[k]} \text{ for all } \gamma \in \Gamma,\ \gamma \notin []^j$$

The system $\mathbf{cGh_p^+}$ is the calculus with connections and the generalized rules $(R \rightarrow^+)$ and $(L\neg\rightarrow^+)$ for the conditional.
Remark: Again the additional formula C is for technical reasons (avoiding empty cedents).

A cut-free derivation of (†) in $\mathbf{cGh_p^+}$

$$
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\dfrac{P^i[j] \Rightarrow P^j[i]}{P^i[j] \Rightarrow P^j[i], (\neg R)^k}
\qquad Q^l[m] \Rightarrow Q^m[l]
}{(P \to Q)^n[m], P^i[m] \Rightarrow Q^m[n,i], (\neg R)^k}
}{(\neg\neg(P \to Q))^n[m], P^i[m] \Rightarrow Q^m[n,i], (\neg R)^k}
}{(\neg(R \to \neg(P \to Q)))^o[m,p], P^i[m] \Rightarrow Q^m[o,i], Q^p[o]}\;(L\neg\to^+)
}{(\neg(R \to \neg(P \to Q)))^o[m], P^i[m] \Rightarrow Q^m[o,i]}
$$

Cut-elimination

Theorem 2

If $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathbf{cGh_p^+}$*, then there is a cut-free derivation of* $\Gamma \Rightarrow \Delta$ *in* $\mathbf{cGh_p^+}$*.*

Sketch of the derivation

Let Δ_1, Π_1 be multisets (possibly empty) of a single mixformula A,

$$(mix)\,\frac{\Gamma \Rightarrow \Delta_0, \Delta_1 \qquad \Pi_1, \Pi_0 \Rightarrow \Lambda}{\Gamma, \Pi_0 \Rightarrow \Delta_0, \Lambda}$$

With the sets of connections at the lower sequent defined as:

$$n' \in []^{\gamma^{l'}} \text{ iff } \begin{cases} n' \in \Delta_0 & \&\ n \in []^{\gamma^l} \text{ at lus, or} \\ n' \in \Lambda & \&\ \exists M^i \in \Delta_1 \text{ at lus } \exists M^j \in \Pi_1 \text{ at rus,} \\ & \quad \text{with } i \in []^{\gamma^l} \&\ n \in []^j; \end{cases}$$

$$n' \in []^{\pi^{l'}} \text{ iff } \quad n' \text{ in } \Lambda \text{ and } n \in []^{\pi^l} \text{ at rus.}$$

Sketch of the derivation

Lemma 3

(i) *If there is a cut- and mixfree derivation D of $A^i, \Gamma \Rightarrow \Delta$ in $\mathbf{cGh_p^+}$, such that $\delta \notin []^i$ in last(D) for all $\delta \in \Delta$, then there is a cut- and mixfree derivation D' of $\Gamma \Rightarrow \Delta$ in $\mathbf{cGh_p^+}$, (depth and connection preserving).*

(ii) *If there is a cut- and mixfree derivation D of $\Gamma \Rightarrow \Delta, A^i$ in $\mathbf{cGh_p^+}$, such that $\gamma \notin []^i$ in last(D) for all $\gamma \in \Gamma$, then there is a cut- and mixfree derivation D' of $\Gamma \Rightarrow \Delta$ in $\mathbf{cGh_p^+}$, (depth and connection preserving).*

Sketch cont.

Lemma 4

1. *If there is a cut- and mixfree derivation D of $\Gamma \Rightarrow \Delta, \perp^j$ in* $\mathbf{cGh_p^+}$*, then there is a cut- and mixfree derivation D' of $\Gamma \Rightarrow \Delta, A^j$ in* $\mathbf{cGh_p^+}$ *(depth and connection preserving);*
2. *if there is a cut- and mixfree derivation D of $(\neg\perp)^j, \Gamma \Rightarrow \Delta$ in* $\mathbf{cGh_p^+}$*, then there is a cut- and mixfree derivation D' of $A^j, \Gamma \Rightarrow \Delta$ in* $\mathbf{cGh_p^+}$*,(depth and connection preserving).*

Sketch cont.

Lemma 5 (Mixelimination)

If there are cut- and mixfree derivations

- *D of $\Gamma \Rightarrow \Delta_0, \Delta_1$ in $\mathbf{cGh}_{\mathbf{p}}^{+}$ and*
- *E of $\Pi_1, \Pi_0 \Rightarrow \Lambda$ in $\mathbf{cGh}_{\mathbf{p}}^{+}$ and Δ_1, Π_1 are (possibly empty) sequences consisting only of a formula M,*

then there is a cut- and mixfree derivation

- *F of $\Gamma, \Pi_0 \Rightarrow \Delta_0, \Lambda$ in $\mathbf{cGh}_{\mathbf{p}}^{+}$,*

such that all the connections at fs(F) are connections at the sequent that would result in an application of mix on fs(D) and fs(E).

Proof by an induction on the grade of the mixformula M with a side induction on the rank of the mix.

$\mathbf{cGh_p^+}$ and $\mathbf{G1h_p}$ are equivalent

In $\mathbf{cGh_p^+}$ theorems are of the form $\top \Rightarrow A$ and antitheorems of the form $A \Rightarrow \bot$.

Theorem 6

The following are equivalent:

1. $\top, \Gamma \Rightarrow \Delta, \bot$ *is derivable in* $\mathbf{cGh_p^+}$*;*
2. $\top, \Gamma \Rightarrow \Delta, \bot$ *is cut-free derivable in* $\mathbf{cGh_p^+}$*;*
3. $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathbf{G1h_p}$*;*
4. $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathbf{G3h_p}$*.*

Establishing the equivalence of the systems

The crucial step is to establish the admissibility of the more general rules $(R \to^{+})$ and $(L\neg\to^{+})$ in the system $\mathbf{cGh_p}$ (with restricted rules) in the presence of (Cut).

- An inductive argument on the depth of the derivation shows that we can basically always work with sequents in which all the formulas without the relevant connections are already in conditional form.

A sequent

$$\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$$

with no connections between Γ_2 and Δ_1 is replaced by

$$\Gamma_1 \Rightarrow \Delta_1, \bigwedge \Gamma_2 \to \bigvee \Delta_2$$

Again we follow the basic strategy of [Kashima and Shimura, 1994].

Summary and outlook

- three different calculi;
- different advantages;
- although $\mathbf{cGh_p^+}$ allows for cut-elimination it is rather tedious to keep track of the connections;
- other applications for connections?
- cut-elimination for full first order HYPE (with constant domains).

Thank you!

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