Logic as the Shadow of Mathematics

Amir Akbar Tabatabai

Bernoulli Institute, University of Groningen

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The Formalistic View

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This is Brouwer's provocative way to challenge the formalistic viewpoint that:

- Mathematics is just a meaningless game with symbols.
- Hence, it is reasonable to start with a logic as the universal rules of reasoning and some axioms as what we agreed upon to develop a theory.

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- Hence, logic is subordinate to mathematics.

Logic is not the foundation of a discourse. It is just its shadow!

The situation is somehow like physics. The world is out there. Physics is just the universal laws of the nature not the rules that nature follows. We can discover the physical laws, but they are subordinate to the nature.

Two Conceptions of Logic

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To do so, we need to formalize constructions at first and then the interpretation of the formulas via this given notion of constructibility.

The Constructions

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More formally, I use IZF which is a system in the usual language of set theory, i.e., $\mathcal{L} = \{\in\}$, using the intuitionistic logic and the Zermelo-Frankel axioms, except for the foundation axiom which is replaced by the following set-induction:

$$\forall x[\forall y \in x A(y) \to A(x)] \to \forall x A(x)$$

and the replacement axiom is replaced by the collection axiom.

Heyting Interpretations

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- $[p]_1$ and $[\bot]_1$ are inhabited,
- $[A \wedge B]_1 = [A]_1 \times [B]_1$,
- $[A \to B]_1 = [B]_1^{[A]_1} = \{f : [A]_1 \to [B]_1\}$,
- $[A \vee B]_1 = [A]_1 + [B]_1$, where $[A]_1 + [B]_1$ is $\{(i, x) \mid (i = 0 \text{ and } x \in [A]_1) \text{ or } (i = 1 \text{ and } x \in [B]_1)\}$.

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- $[p]_0 \subseteq [p]_1$, for any atomic formula p and $[\bot]_0 = \emptyset$,
- $[A \wedge B]_0 = [A]_0 \times [B]_0$,
- $[A \to B]_0 = \{f \in [A \to B]_1 \mid \forall x \in [A]_0 \; f(x) \in [B]_0\}$,
- $[A \vee B]_0 = [A]_0 + [B]_0$.

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To show that the function is an actual construction, we must show that it maps any element in $[p \wedge q]_0$ to an element in $[p]_0$ which is trivially true.

Some Examples

Example

- $p \to (q \to p)$, by the map $x \mapsto \lambda y.x$,
- $p \to p \vee q$ by the map $x \mapsto (0, x)$,
- $q \to p \vee q$ by the map $x \mapsto (1, x)$
- $(p \to q) \to [(q \to r) \to (p \to r)]$, by the map $f \mapsto \lambda g.(f \circ g)$,
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What is the Heyting theory of constructive sets?

Having the previous example, it is easy to see that $\mathbf{T}^H \supseteq \mathrm{IPC}$. Does the equality hold?

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Now, we must define $F : [\neg p \to (q \vee r)]_1 \to [(\neg p \to q)]_1 + [(\neg p \to r)]_1$.

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- Apply f on a. Hence, $f(a) \in [q]_1 + [r]_1$,
- Then, $F(f) = (\pi_0(f(a)), \lambda x.\pi_1(f(a))) \in [(\neg p \to q)]_1 + [(\neg p \to r)]_1$.

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Now, assume that $f : [\neg p \to q \vee r]_0$. Then,

- $\pi_0(f(a))$ is either zero or one.
- If it is zero, then $\lambda x.\pi_1(f(a))$ is in $[\neg p \to q]_0$, because if $x \in [\neg p]_0$, then $a \in [\neg p]_0$, which implies that $f(a) \in [q \vee r]_0$.
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Remark

The core ideas behind the Heyting validity of the axiom KP are:

- The explicit information encoded in any proof of a disjunction,
- The fact that $\neg A$ has the following property: If $a \in [\neg A]_0$, then $[\neg A]_0 = [\neg A]_1$.

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To provide a machinery to prove this conjecture and for some other philosophical reasons, it is reasonable to also focus on different families of interpretations.

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- If an interpretation is proof-irrelevant, then $[A]_1$ is a singleton, for any $\vee$-free formula A. However, when the disjunction appears, a proof can contain some information (at least identifying the provable disjunct) and hence cannot be proof-irrelevant.

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- An interpretation is Markov and proof-irrelevant iff it is Kolmogorov with singleton $[p]_1$ iff $[p]_1 = \{*\}$ and the sentence $(* \in [p]_0)$ is negative.

The Theory and the Logic of a Mathematical World

Definition

Let $\mathcal{C}$ be a definable class of Heyting interpretations.

- By the $\mathcal{C}$-Heyting theory of constructive sets, denoted by $\mathbf{T}^H_{\mathcal{C}}$, we mean the set of all propositional formulas A such that $\forall [-] \in \mathcal{C}\ \exists x \in [A]_0$.

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For the definable classes of Markov, Kolmogorov and proof-irrelevant interpretations, we use M, K and PI, respectively.

Some Heyting Theories and Heyting Logics

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Theorem

- *Proof-irrelevant (Theory):* $\mathbf{T}^H_{PI} = \mathrm{INP} = \mathrm{IPC} + \{(A \to (B \vee C)) \to ((A \to B) \vee (A \to C)) \mid A \text{ is } \vee\text{-free}\}$.
- *Proof-irrelevant (Logic):* Conjecture: $\mathbf{L}^H_{PI} = \mathrm{KP}$.
- *Markov and Proof-irrelevant (Theory):* $\mathbf{T}^H_{MPI} = \mathrm{KP}^n = \mathrm{KP} + \{\neg\neg p \to p \mid p \text{ is an atom}\}$.
- *Markov and Proof-irrelevant (Logic):* $\mathbf{L}^H_{MPI} = \mathrm{ML}$.
- *Kolmogorov:* $\mathbf{L}^H_K = \mathrm{ML}$.

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$[A \vee B]_1 = \|[A]_1 + [B]_1\|$, where $\| - \|$ is the propositional truncation, i.e., $\|X\| = \{x \in \{0\} \mid \exists y \in X\}$ and
$[A \vee B]_0 = \{x \in \{0\} \mid \exists y \in [A]_0 \vee \exists y \in [B]_0\}$.

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$[A \vee B]_1 = \|[A]_1 + [B]_1\|$, where $\| - \|$ is the propositional truncation, i.e., $\|X\| = \{x \in \{0\} \mid \exists y \in X\}$ and
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Given a construction of a disjunction:

- **Heyting:** Total decidability of which disjunct is provable and a direct access to the corresponding proof.
- **Brouwer**: No non-trivial information about the provable disjunct or its corresponding proof.

The Characterization of Brouwerian Logic

Theorem

$\mathbf{T}^B = \mathsf{IPC}$.

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Proof.

The soundness is easy! For the completeness, let τ be a set-theoretical substitution and set $[p]_1 = [\bot]_1 = \{0\}$ and $[p]_0 = \{x \in \{0\} \mid \tau(p)\}$. It is easy to see that any $[A]_1$ has exactly one canonical element. Call it θ_A. Then it is also easy to see that $\theta_A \in [A]_0$ iff $\tau(A)$. Therefore, $\mathsf{IZF} \vdash \tau(A)$, for any set-theoretical substitution. By the recent **Passman**'s beautiful theorem, we have $\mathsf{IPC} \vdash A$. □

Other Brouwerian Corollaries

Remark

Note that Brouwer's interpretation is just the *truth-value computation* and hence the Brouwerian logic of a theory is its propositional logic in the usual sense.

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The followings are the consequences of the previous theorem:

Corollary

- *Proof-irrelevant:* $\mathbf{T}^B_{PI} = \mathrm{IPC}$.
- *Markov and Proof-irrelevant (Theory):* $\mathbf{T}^B_{MPI} = \mathrm{IPC}^n = \mathrm{IPC} + \{\neg\neg p \to p \mid p \text{ is an atom}\}$.
- *Markov and Proof-irrelevant (Logic):* $\mathbf{L}^B_{MPI} = \mathrm{IPC}$.
- *Kolmogorov:* $\mathbf{L}^B_K = \mathrm{IPC}$.

BHK Interpretations and their Theories and Logics

Heyting's Interpretation	Theory	Logic
without conditions	above KP?	above KP?
Proof-irrelevant	INP	above KP?
Markov (up to proof-irrelevancy)	KP^n	ML
Kolmogorov	?	ML

Brouwer's Interpretation	Theory	Logic
without conditions	IPC	IPC
Proof-irrelevant	IPC	IPC
Markov (up to proof-irrelevancy)	IPC^n	IPC
Kolmogorov	?	IPC

Thank you for your attention!