What is Formalisation?

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Talk outline

Formalisation

- As the suffix indicates, formalisation is a process with inputs and outputs.
- More precisely, formalisation is a map that takes sentences (a placeholder notion) of a language such as English to sentences of a language such as that of propositional logic (PL) or first-order logic (FOL).
- Call the informal language E, the formal language or logic $\mathcal{L}$ and the formalisation map $Form : Sen(E) \to Sen(\mathcal{L})$.
- What makes such a map a formalisation? (Or if you prefer: what makes it a good formalisation?)

Acceptable/good formalisations...

Compare these formalisations:

$Form_{\mathrm{PL}}$ (Fido is a dog) $= p$
$Form_{\mathrm{PL}}$ (There's a dog) $= q$
$Form_{\mathrm{FOL}}$ (Fido is a dog) $= Fa$
$Form_{\mathrm{FOL}}$ (There's a dog) $= \exists x Fx$

...versus unacceptable/bad formalisations

with the following formalisations:

$Form_{\text{PL}}$ (Fido is a dog) $= p \leftrightarrow (q \vee r)$
$Form_{\text{PL}}$ (There's a dog) $= q \wedge r$
$Form_{\text{FOL}}$ (Fido is a dog) $= \forall x \forall y (Fx \leftrightarrow Gy)$
$Form_{\text{FOL}}$ (There's a dog) $= \forall x (x = x)$

Talk outline

Three criteria

- Capturing Implication
- Respecting Grammar
- Semantic Proximity

Capturing Implication

- Some E-sentences stand in the relation of implying another; for example, 'Fido is a dog' logically implies 'There's a dog' but does not logically imply 'There's a cat'.
- The better the formalisation the more accurately it captures these implicational relations.

Respecting Grammar

- A formalisation should respect the grammatical form of an E-sentence as much as possible.
- Contrast the propositional formalisation of ‘The cat sat on the mat’, which is the sentence letter p, with the propositional formalisation of ‘The cat sat on the mat, and it’s raining or it’s not raining’, which is $p \wedge (q \vee \neg q)$.

Semantic Proximity

- This criterion enjoins us to formalise E-sentence s as formal sentence σ if σ may be interpreted so as to be as close in meaning to s as possible.
- As Benson Mates put it:

  > *...to formulate precise and workable rules for symbolizing sentences of the natural language is a hopeless task. In the more complicated cases, at least we are reduced to giving the empty-sounding advice: ask yourself what the natural language sentence means, and then try to find a sentence of [the formal language] $\mathfrak{L}$ which, relative to the given interpretation, has as nearly as possible the same meaning. (Mates 1972, p. 84)*

Talk outline

PL and FOL matched against the criteria

- Judged against these criteria, which does better, PL or FOL?
- Counterintuitive verdict: it's a tie—they do equally well!
- Some results from two papers of mine: 'Capturing Consequence' (RSL 2019) and 'Propositionalism' (JPhil 2021).

FOL wins hands down?

Argument	PL-Formalisation	FOL-Formalisation
Bruin is a bear ∴ There's a bear.	p $\therefore q$	Bb $\therefore \exists x Bx$
Zeno is a tortoise All tortoises are toothless ∴ Zeno is toothless	p q $\therefore r$	To $\forall x(Tx \rightarrow Hx)$ $\therefore Ho$

PL strikes back

Argument	Propositional Formalisation
Bruin is a bear ∴ There's a bear	p $\therefore p \vee q$
Zeno is a tortoise All tortoises are toothless ∴ Zeno is toothless	p $(p \rightarrow q) \wedge (r \rightarrow s)$ $\therefore q$

Can this trick be played more generally?

- Answer: yes.
- Proof in 'Capturing Consequence'.
  (Unknown to me at the time, Jeřábek (2012) beat me to some of the underlying formal results. Though he used a very different proof and his interests were also very different.)

- Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are two logics, with respective sets of sentences $Sen(\mathcal{L}_1)$, $Sen(\mathcal{L}_2)$ and respective consequence relations $\vDash_{\mathcal{L}_1}, \vDash_{\mathcal{L}_2}$.
- The map $j : Sen(\mathcal{L}_1) \to Sen(\mathcal{L}_2)$ is a *conservative translation* (or a *faithful* translation) just when, for all $\Gamma \subseteq Sen(\mathcal{L}_1)$ and $\delta \in Sen(\mathcal{L}_1)$,
$$\Gamma \vDash_{\mathcal{L}_1} \delta \text{ if and only if } j(\Gamma) \vDash_{\mathcal{L}_2} j(\delta)$$
- A *bijective conservative translation* or *consequence isomorphism* is a conservative translation $j : Sen(\mathcal{L}_1) \to Sen(\mathcal{L}_2)$ that is also a bijection.
- The result is that *countable* PL and FOL are consequence-isomorphic.
- Result generalises.
- Corollary for first criterion: PL and FOL satisfy it equally well.

What about the other two criteria?

- ‘Propositionalism’ extended the above result.
- For any given FOL-formalisation, there is a PL one that not only matches first-order logic implicationally but also, *by the propositional logician’s lights*, meets the grammatical constraint.
- Semantic proximity also does not discriminate between PL and FOL, at least if cashed out in terms of truth-conditions.

Talk outline

Formalisation as a Mapping

Three Criteria

A case study: PL vs FOL

The Problem of the Missing Criterion

Formalisation as a Map

What is the missing criterion?

- Either live with this result or assume there is a missing criterion (at least).
- Against the first option: think of the great formalising drives of Frege, Hilbert and others.
- So what might the missing criterion be?
- A suggestion: it's epistemic.
- More precisely: $\mathcal{L}_1$-formalisations are superior to $\mathcal{L}_2$-formalisations if the former generally allow us to discern facts about validity better than the latter.

Talk outline

The nature of formalisation

- Suppose formalisation is epistemically constrained. Does that mean logical consequence is also epistemically constrained?
- Consider the biconditional:

  For all statements s of E and sets of statements S of E, S logically entails that s iff $Form_{\mathcal{L}}(S) \vDash_{\mathcal{L}} Form_{\mathcal{L}}(s)$.

- First picture: RHS has 'priority'. But do $\mathcal{L}$-facts really constitute the relation of logical consequence among E-sentences?
- Second picture: RHS captures LHS but does not have 'priority'. It captures rather than constitutes.

The Map Analogy

- Relationship of logic to logical consequence is akin to the relationship of a map to some territory it describes.
- Under this analogy, the correct logic for capturing logical consequence is akin to a perfect map.
- Suppose for example that $\mathcal{M}$ is a perfect map of the Netherlands. So it describes orientation facts such as:

  Utrecht is south of Amsterdam iff $\mathcal{M}$ depicts Utrecht as south of Amsterdam.
- A perfect map does more than encode the orientation of some places vis-à-vis others; it does so in a way that is readable by the map's users.
- The suggestion is that this is analogous to logic's encoding of implication relations.
- Which helps us see why the fourth criterion is needed.