Workshop on Proofs and Formalization in Logic, Mathematics and Philosophy
@Utrecht, the Netherlands

A Formalisation of
Crispin Wright's Strict Finitistic First-order Logic

Takahiro Yamada*
Utrecht University (PhD Student)
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* g.yamadatakahiro@gmail.com

What is Strict Finitism?

- Another constructive view of mathematics.
  - Based on practical constructibility (& verifiability).
    - Constructive: Replace “in principle” of intuitionism by “in practice”.

“A number is constructible in principle iff it is constructible in practice with some finite extension of cognitive resources.”

- Finitistic: $\mathbb{N}$ vs as much as you can actually construct (represent).

Topics

Today's Aims

- Show Wright's informal SF argument.
- Provide a formal logic based on Wright's sketch, similar to intuitionistic logic.
  - ➢ In the classical metatheory.
  - ➢ A complete pair of the semantics & a proof system.
- Present some informal notions formalised.
  - ➢ Incl. *that* relation with intuitionism.
  - ➢ To explicate the philosophical standpoint.

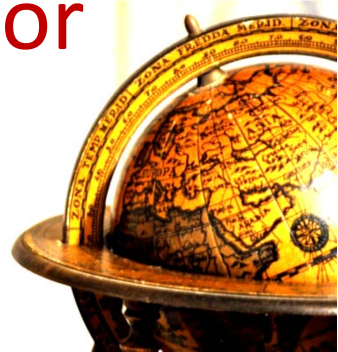
SF: **S**trict **F**initism (**F**initistic)

1. Wright’s Informal Argument

Concepts

Wright's SF Metatheory

- Practical possibilities (e.g. constructibility) satisfy…
  - (Basis) There is a starting point: e.g. 0 is constructible.
  - (Tolerance) If something is constructible, then anything `adjacent' (e.g. successor) is constructible.
  - (Boundedness) There is an unconstructible upper bound to those constructible.
  - (Decidability) Anything is either constructible or unconstructible.
- Apply also to verifiability, representablity etc.

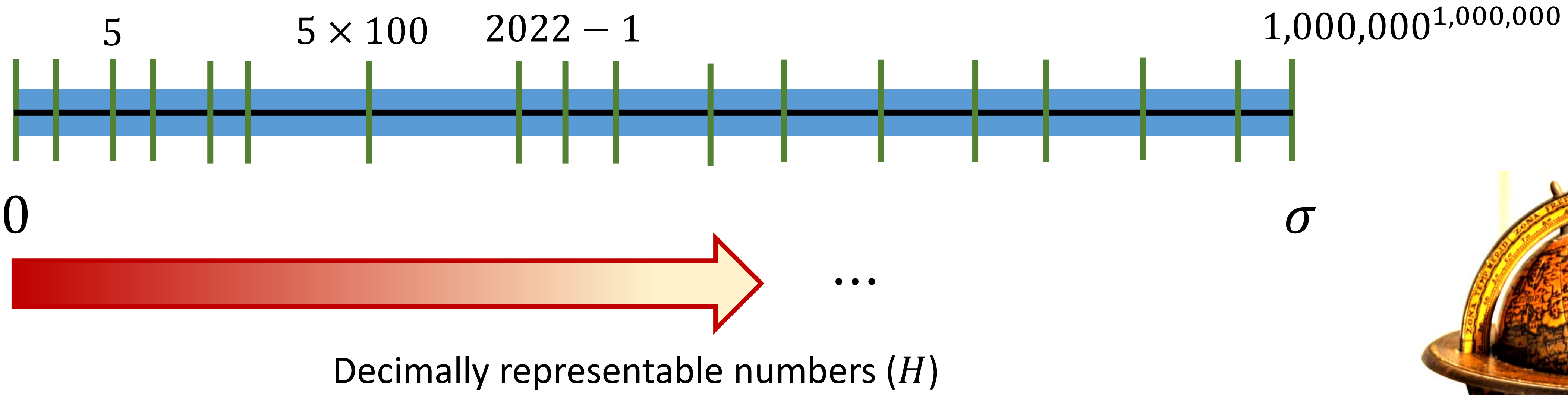


The Setting

- Two practical possibility predicates.
  - $H(x)$: x is decimally representable.
  - $H'(x)$: x is representable in some notation.
- Two functions.
  - $p(x)$: x's predecessor.
  - $s(x)$: x's successor.
- With tolerance
  - $\forall x[H(x) \rightarrow H(p(x)) \wedge H(s(x))]$.
  - $\forall x[H'(x) \rightarrow H'(p(x)) \wedge H'(s(x))]$.
- An object.
  - σ: with $H'(\sigma)$ and $\neg H(\sigma)$: e.g. $1{,}000{,}000^{1{,}000{,}000}$.

The Setting

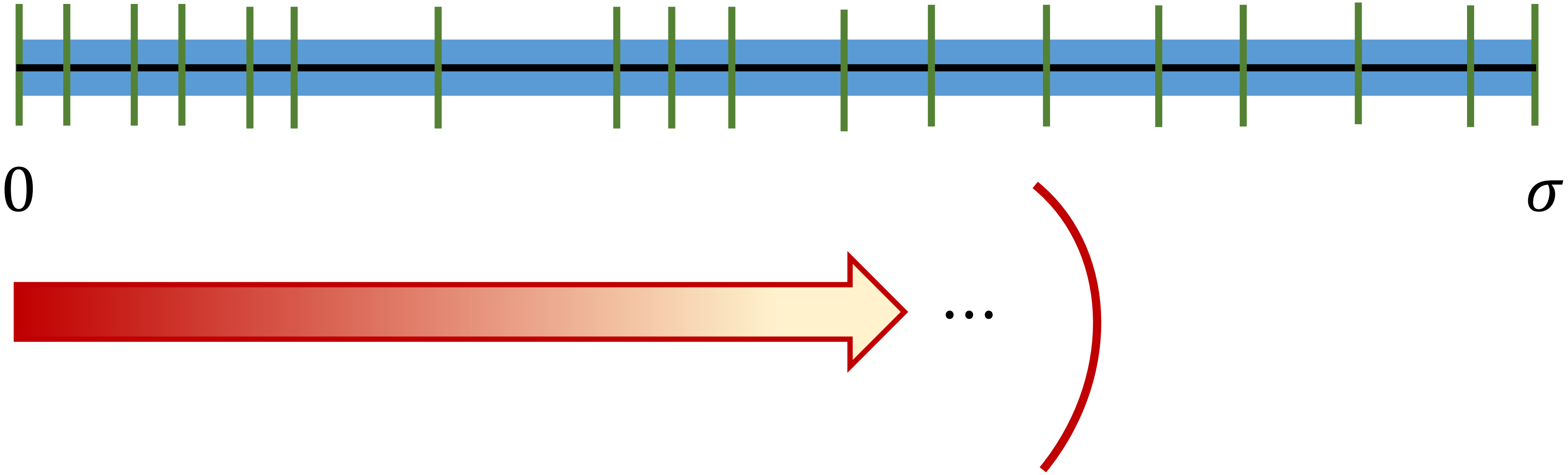
- Two collections of numbers.
  - ➢ $\Sigma := \{n | H'(n) \land 0 \leq n \leq \sigma\}$ …… Representable numbers (H')
  - ➢ $\Sigma^{-} := \Sigma \backslash \{0, \sigma\}$.



Argument

(My) Observation

- H's extension is closed under s: $H(n) \Rightarrow H(s(n))$.

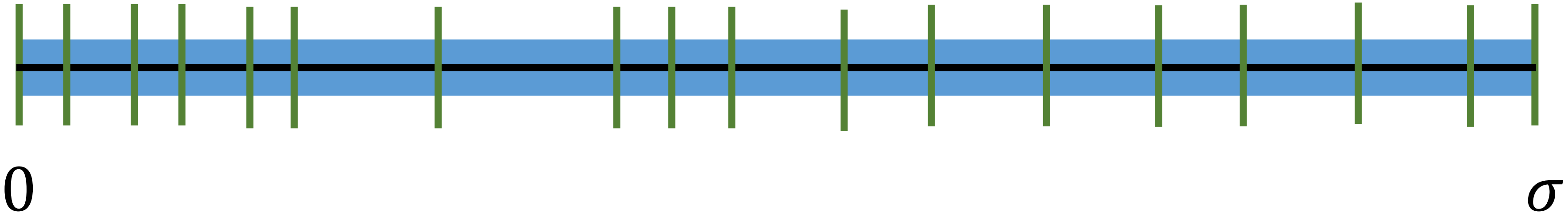


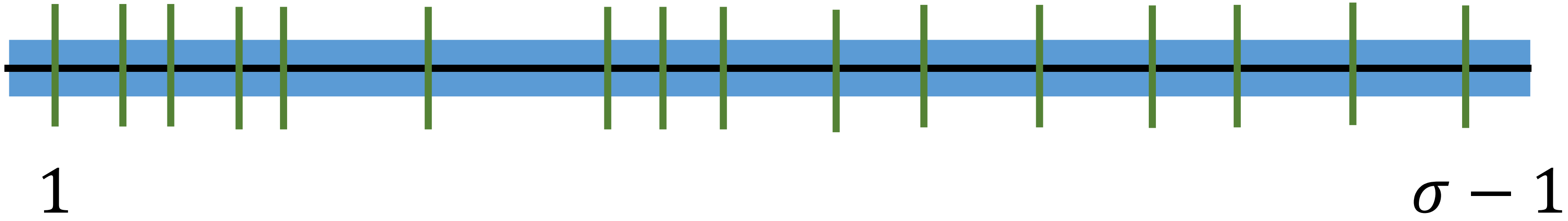
- Its complement is closed under p: $\neg H(n) \Rightarrow \neg H(p(n))$.
  - ➢ *As though* $H(n)$ means that n is a "standard number", as opposed to a "nonstandard" one.

Argument

The Claim

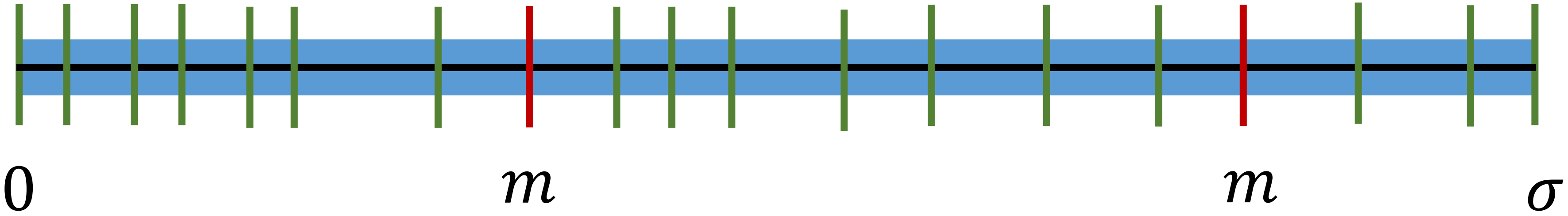
- There is a bijection between Σ and Σ^{-}.



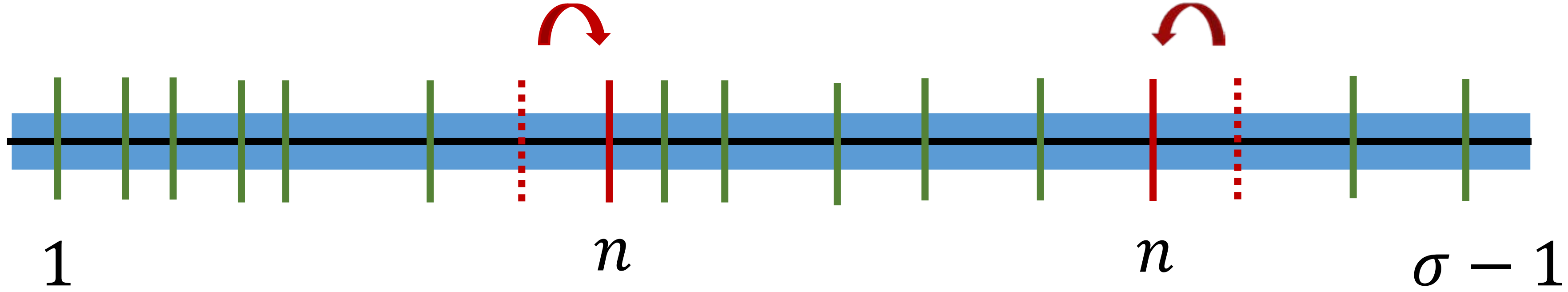


The Argument

- Define a collection $f \subseteq \Sigma \times \Sigma^-$ of pairs by



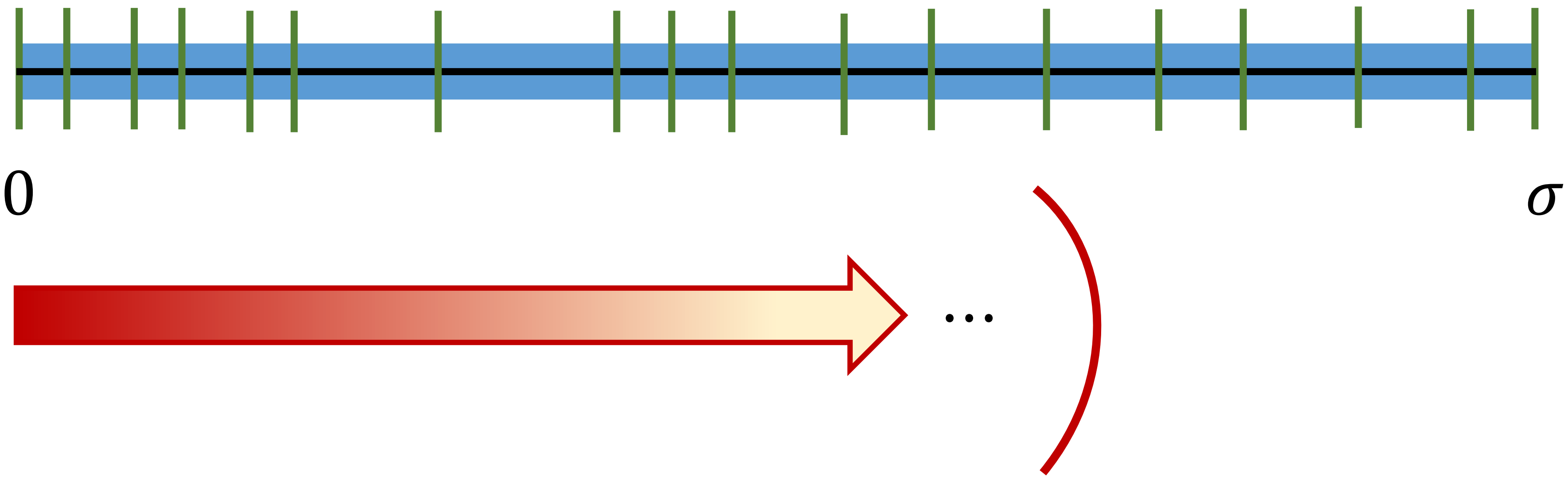
$$(m, n) \in f \iff \big(H(m) \land n = s(m)\big) \lor \big(\neg H(m) \land n = p(m)\big).$$



Argument

Functionality

- $\forall m \in \Sigma\ \exists!\, n \in \Sigma^{-}\ [(m, n) \in f]$.

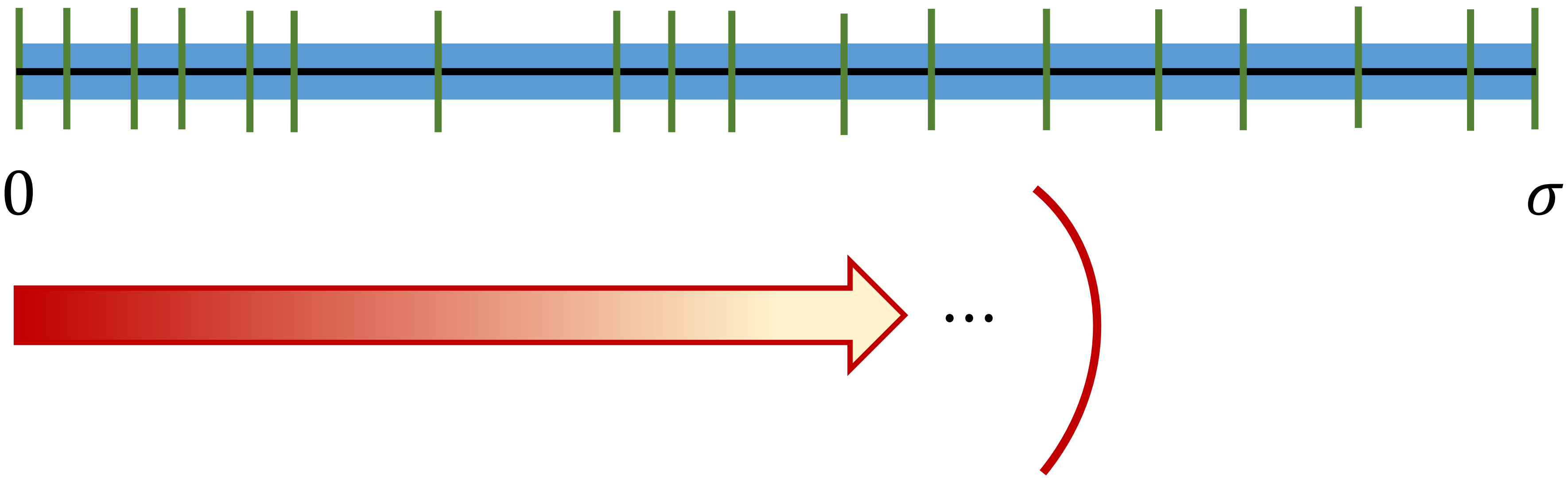


- Case distinction: $H(m)$ or $\neg H(m)$.
- The uniqueness comes from uniqueness of s and p.

Argument

Injectivity

- $\forall m, m' \in \Sigma\, [f(m) = f(m') \Rightarrow m = m']$.

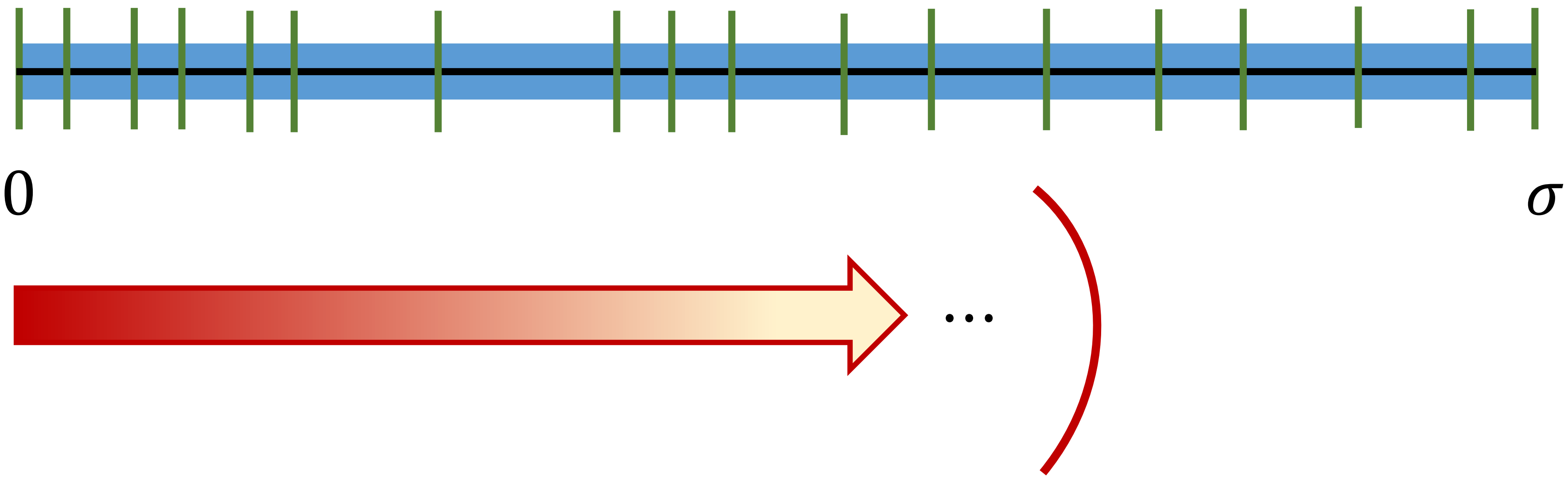


- Case distinction: $H(m)$ or $\neg H(m)$.
- Use uniqueness of s and p.

Argument

Surjectivity

- $\forall n \in \Sigma^{-}\ \exists m \in \Sigma\,[f(m) = n]$.



- Case distinction: $H(n)$ or $\neg H(n)$.

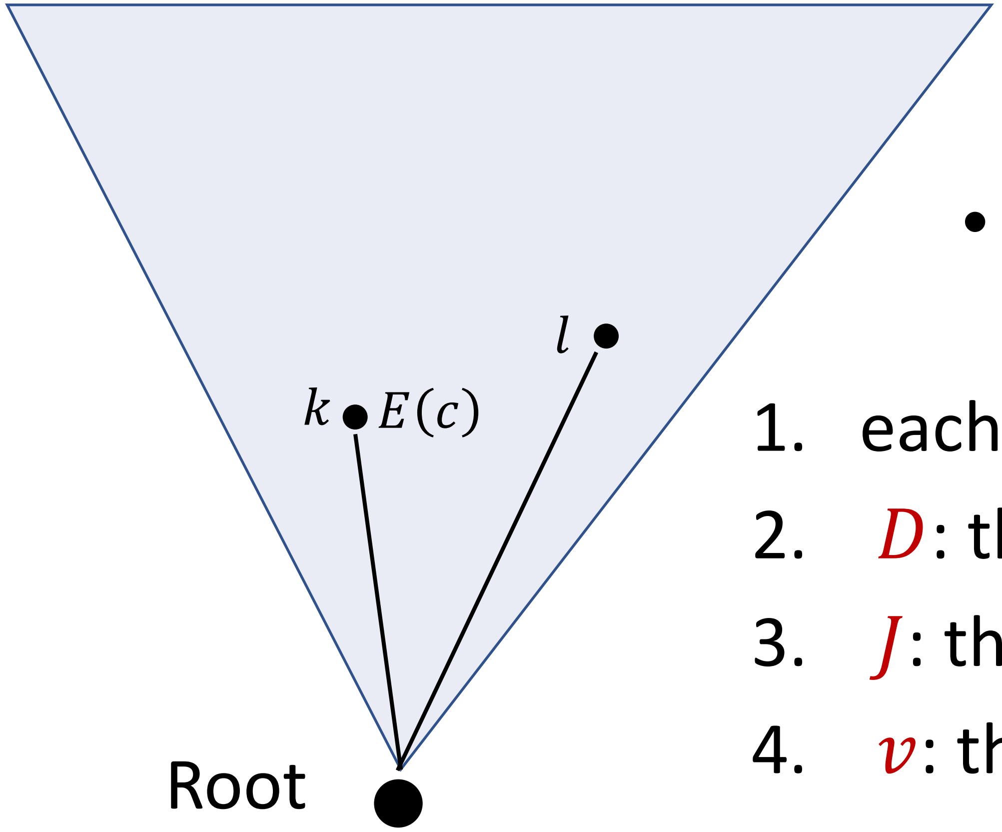
2. Formal Logic: Semantics

Topics

SF to Classical Metatheory

- Wright: "strict finitistic trees" and forcing conditions in his SF metatheory.
- Interpret into the classical metatheory.
  - Make SF inferences intelligible to us.
  - Use classical principles: induction & LEM.
  - Formalise SF principles.
- A semantics similar to **IQCE**, with the "existence" predicate E.

Language & Models



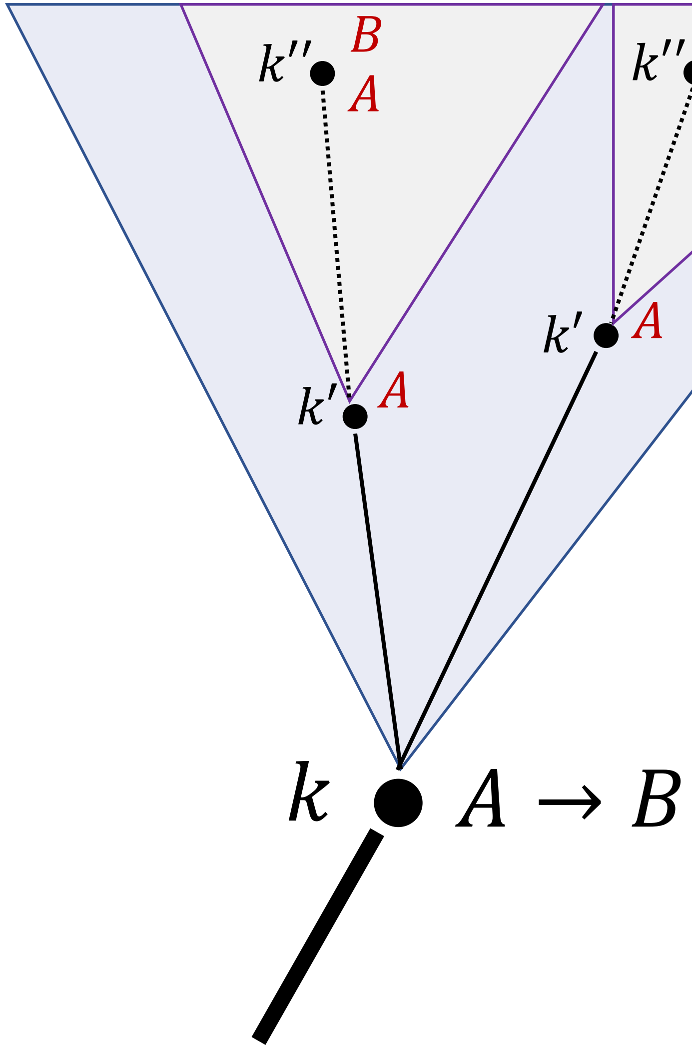
- The `existence' predicate E: "constructed" or "available".
- Rooted tree-like intuitionistic models $\langle K, \leq, D, J, v\rangle$ such that…

1. each branch is at most countably long;
2. D: the constant domain.
3. J: the interpretation function.
4. v: the valuation function.

➢ Represents all possible histories of the agent's actual verification, from our perspective.

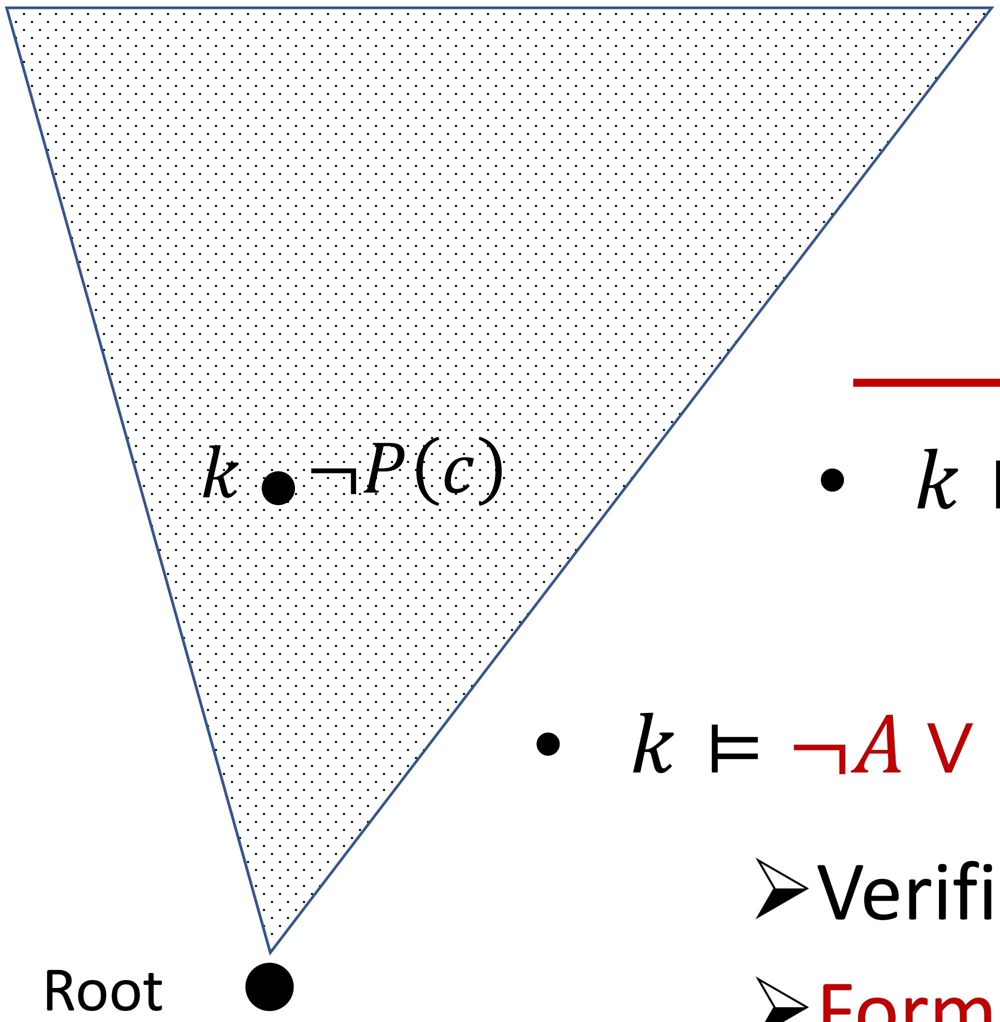
➢ Strictness: $k \vDash P(c) \Longrightarrow k \vDash E(c), k \vDash E\big(f(c)\big) \Longrightarrow k \vDash E(c)$.

Strict Finitistic Implication



- "Practical implication".
- $k \vDash A \to B$ iff, for any $k' \geq k$, if $k' \vDash A$, then there is a $k'' \geq k'$ such that $k'' \vDash B$.
- "B comes after A sooner or later".
- Intuitionistic implication with time-gap.
- $\sim A := A \to \bot$
  - Intuitionistic, local negation.

Global Negation

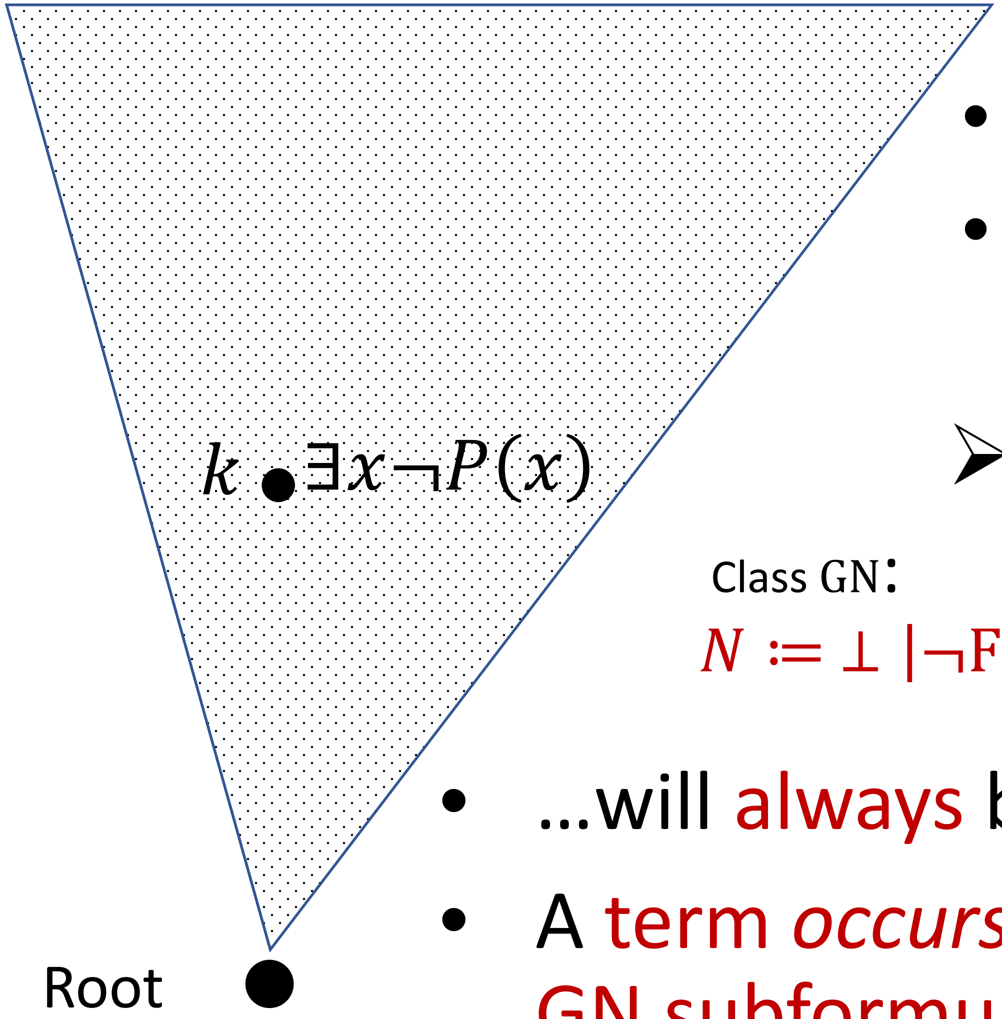


- $k \vDash \neg A$ iff $l \nvDash A$ for all nodes l.
  - Practically unverifiable.
  - If somewhere, then everywhere.

- $k \vDash \neg\neg A$ iff $l \vDash A$ for some l.
  - Practically verifiable.
- $k \vDash \neg A \vee \neg\neg A$ for all k: "Weak LEM" is valid.
  - Verifiability is decidable.
  - Formalisation of (decidability).

Ideas

2 Modes of Quantification



- $P(c)$ refers to a constructed objects.
- $\neg P(c)$ refers to *an object* in the scope of discourse.

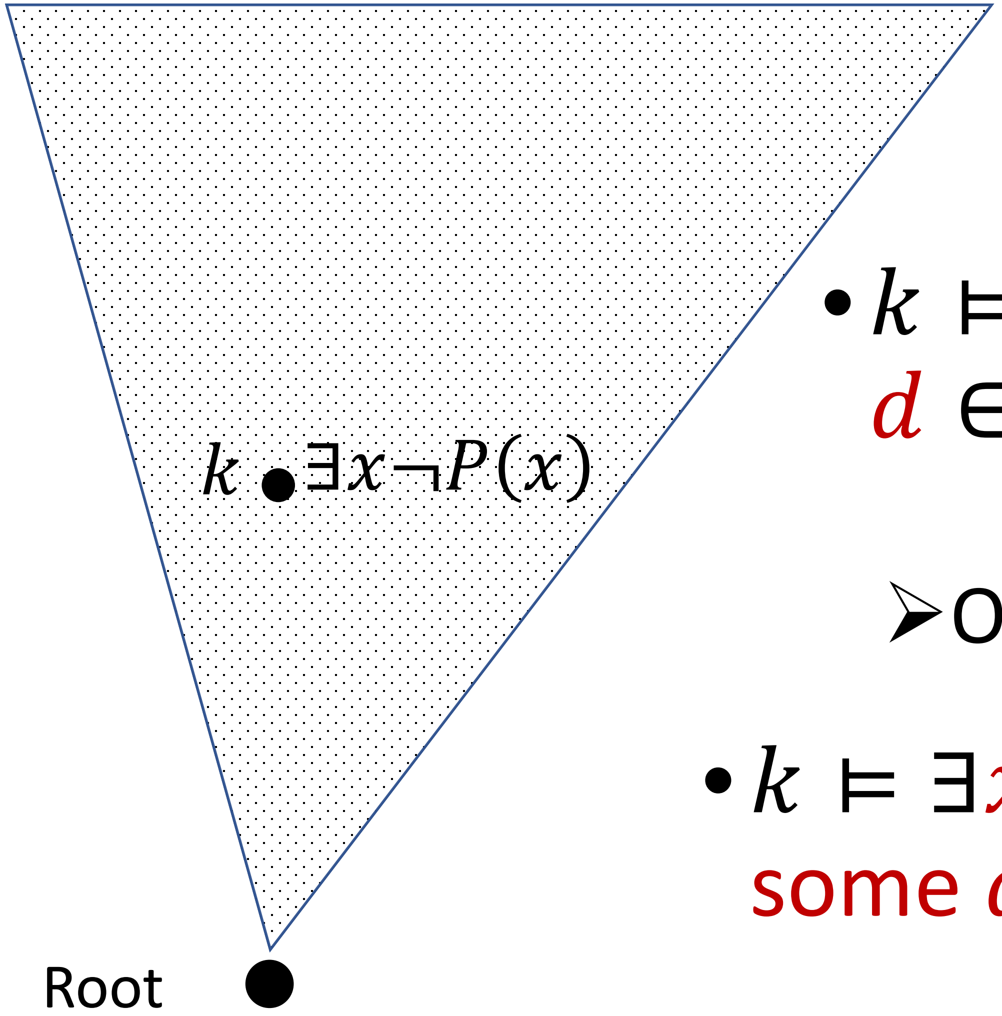
➢ "Local" & "global" quantification.

Class GN:

$N \coloneqq \bot \mid \neg \mathrm{Form} \mid N \wedge N \mid N \vee N \mid N \to N \mid \forall x N \mid \exists x N$

- …will always be global ("**G**lobal **N**egative").
- A term *occurs* in A *globally* if it occurs in a GN subformula of A.

Global & Local ∃



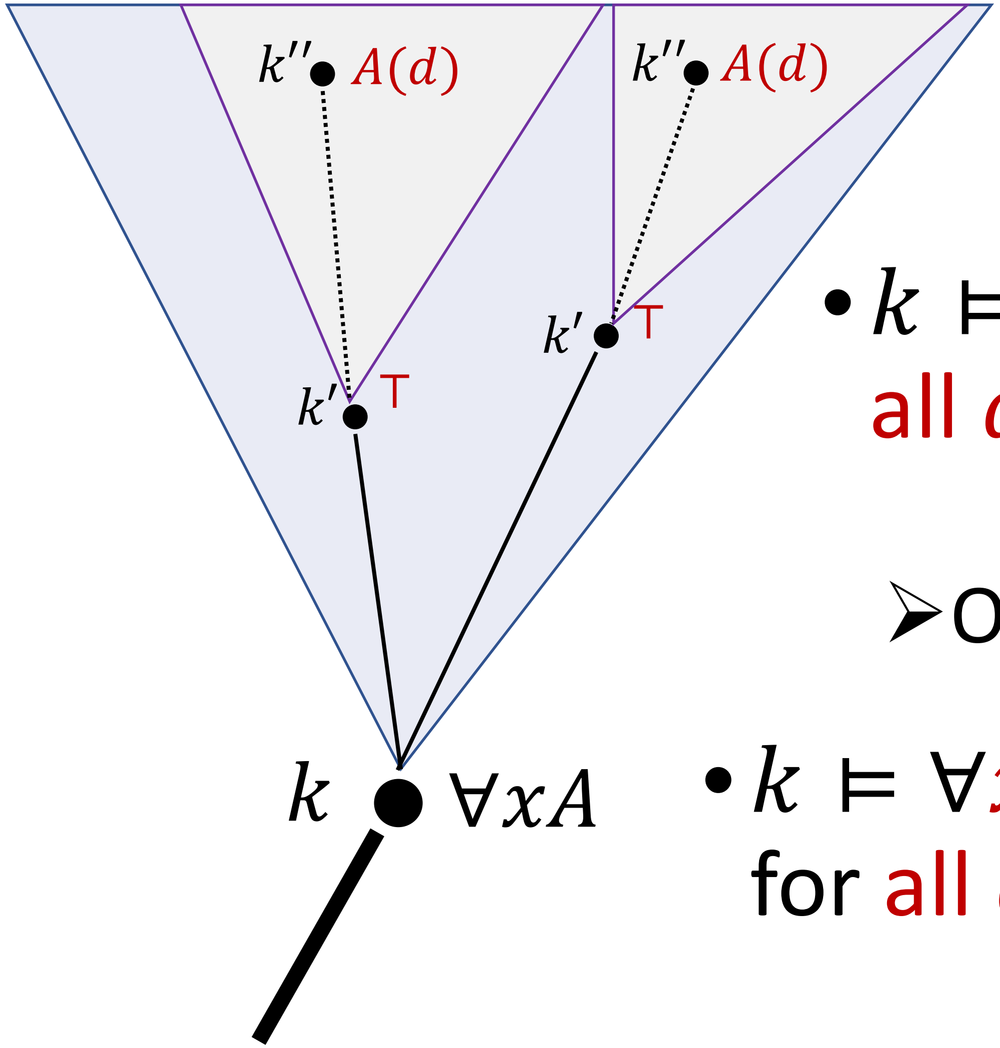
➢ For x occurring only globally.

- $k \vDash \exists x A$ iff $k \vDash A[\overline{d}/x]$ for some $d \in D$.

➢ Otherwise…

- $k \vDash \exists x A$ iff $k \vDash E(\overline{d}) \wedge A[\overline{d}/x]$ for some $d \in D$.

Global & Local $\forall$



- For x occurring only globally.
  - $k \vDash \forall x A$ iff $k \vDash \top \to A[\bar{d}/x]$ for all $d \in D$.
- Otherwise…
  - $k \vDash \forall x A$ iff $k \vDash E(\bar{d}) \to A[\bar{d}/x]$ for all $d \in D$.

Validity

- $\mathcal{W}$: the class of all models.
- A is *valid in* $W \in \mathcal{W}$ ($\vDash_W^V A$) if forced at all nodes in W.
- A is *valid in* $\mathcal{W}$ ($\vDash_{\mathcal{W}}^V A$) if forced in all $W \in \mathcal{W}$.
- A is a *semantic consequence* of Γ in W ($\Gamma \vDash_W^V A$) if for all node k, $k \vDash B$ for all $B \in \Gamma$ implies $k \vDash A$.
  - $\Gamma \vDash_{\mathcal{W}}^V A$ is likewise.

Valid Formulas

- Hold: $\neg A \vee \neg\neg A$, $\sim\sim A \to A$, $\big((A \to B) \to A\big) \to A$.

- Fail: $A \vee \neg A$, $\neg\neg A \to A$, MP ($\vDash^V A \to B \ \&\ \vDash^V A \Longrightarrow \vDash^V B$.)

3. Formal Logic: Proof System

Natural Deduction **NSF**

- (∧) & (∨): Classical. $\dfrac{}{\top}\ \top I \qquad \dfrac{\bot}{A}\ \bot E$
- (Strictness): $\dfrac{P(c)}{E(c)}\ \mathrm{STR}_1 \qquad \dfrac{E(f(c))}{E(c)}\ \mathrm{STR}_2$
- (Stability): $\dfrac{\sim\sim S}{S}\ \mathrm{ST}$

Class GT: $S := \top | N | S \wedge S | S \vee N | N \vee S | \mathrm{Form} \to \mathrm{Form} | \forall x \mathrm{Form}$

$N \in \mathrm{GN}$

- A is *stable* in $W \in \mathcal{W}$ if $\sim\sim A \vDash_W^V A$.
- For all $A \in \mathrm{ST}$, $\sim\sim A \vDash_W^V A$.

Natural Deduction **NSF**

$$\frac{\begin{matrix}[A]\\ \vdots \\ \sim\sim B\end{matrix}}{A \to B}\ \to\!\text{I} \qquad \frac{A \to B \quad A}{\sim\sim B}\ \to\!\text{E}$$

$$\frac{\sim N}{\neg N}\ \neg\text{I}_1 \qquad \frac{\begin{matrix}\color{red}{\text{GN}}\\ \vdots \\ \sim A\end{matrix}}{\neg A}\ \neg\text{I}_2$$

$$\frac{\sim A \quad A}{\bot}\ \sim\!\text{E} \qquad \frac{\neg A \quad A}{\bot}\ \neg\text{E}$$

$$\frac{\sim\sim A[y/x]}{\forall x A}\ \forall\text{-glo I} \qquad \frac{\begin{matrix}[E(x)]\\ \vdots \\ \sim\sim A[y/x]\end{matrix}}{\forall x A}\ \forall\text{-nglo I}$$

$$\frac{\forall x A}{\sim\sim A[t/x]}\ \forall\text{-glo E} \qquad \frac{\forall x A \quad E(t)}{\sim\sim A[t/x]}\ \forall\text{-nglo E}$$

$$\frac{A[t/x]}{\exists x A}\ \exists\text{-glo I} \qquad \frac{A[t/x] \quad E(t)}{\exists x A}\ \exists\text{-glo I}$$

$$\frac{\exists x A \quad \begin{matrix}[A[y/x]]\\ \vdots \\ C\end{matrix}}{C}\ \exists\text{-glo E} \qquad \frac{\exists x A \quad \begin{matrix}[A[y/x]]\ [E(x)]\\ \vdots \\ C\end{matrix}}{C}\ \exists\text{-nglo E}$$

Theorems

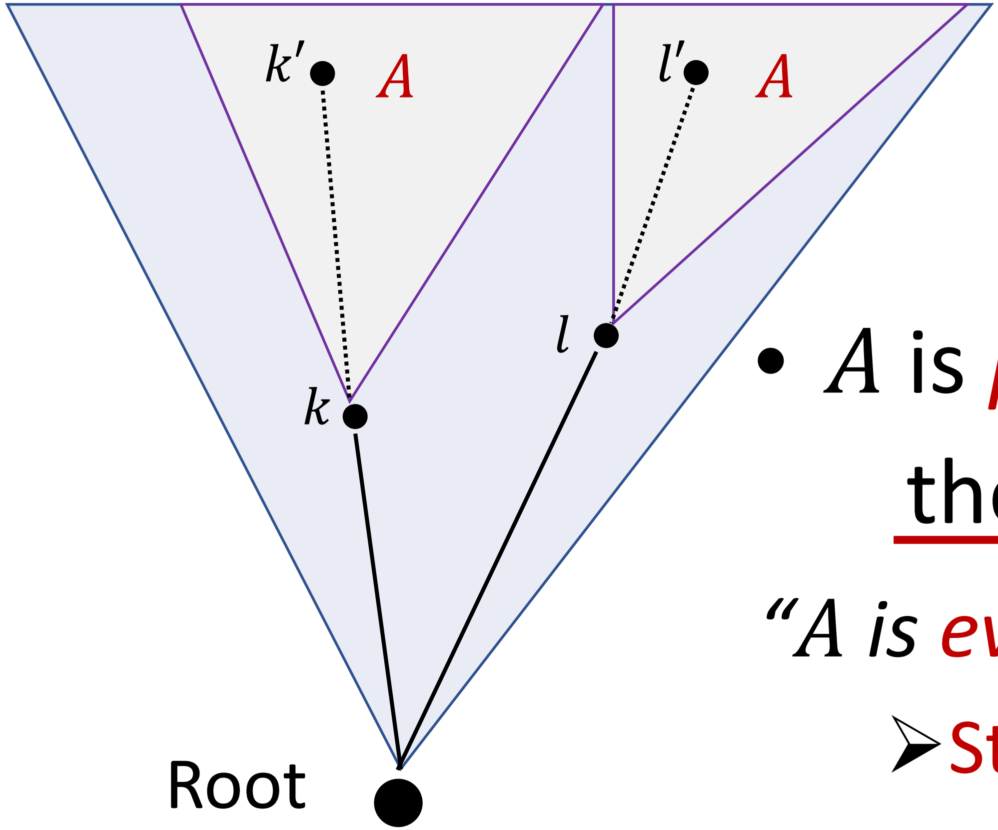
Claim: Soundness & Completeness

- Soundness: $\Gamma \vdash_{\mathbf{NSF}} A$ implies $\Gamma \vDash_{\mathcal{W}}^{V} A$.
  - ➢Routine.

- Completeness: $\Gamma \vDash_{\mathcal{W}}^{V} A$ implies $\Gamma \vdash_{\mathbf{NSF}} A$.
  - ➢Complicated, but a usual Henkin-style proof.

4. Prevalence: A “Rejected” Principle

Prevalence: Strong Verifiability



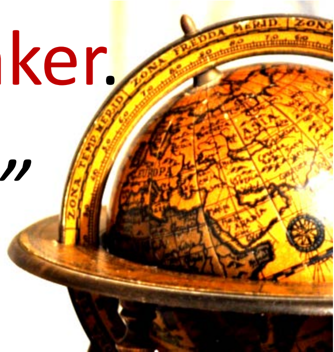
"$\vDash^P A$"

- A is *prevalent* if for any node k, there is a $k \geq k'$ such that $k' \vDash A$.

"A is eventually verified, in any case."

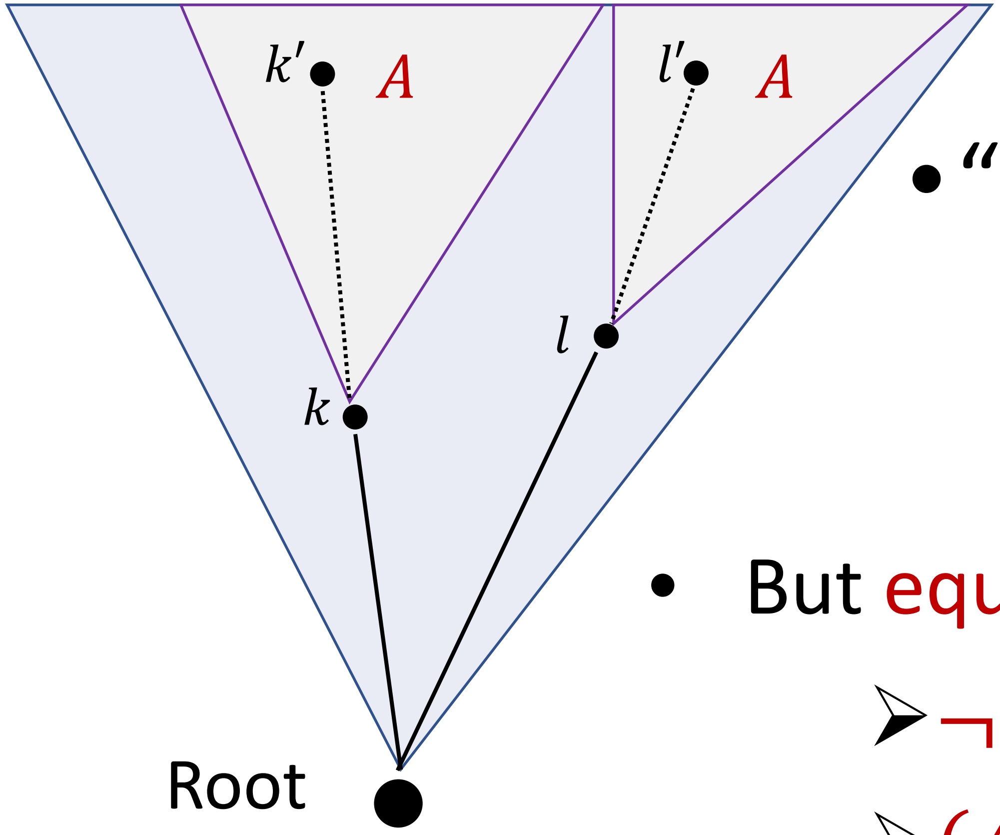
- Stronger practical verifiability:
  - Satisfiability ($\neg\neg A$) is weaker.

"A is verified, in some case."



Critical

"Formula Prevalence" Principle

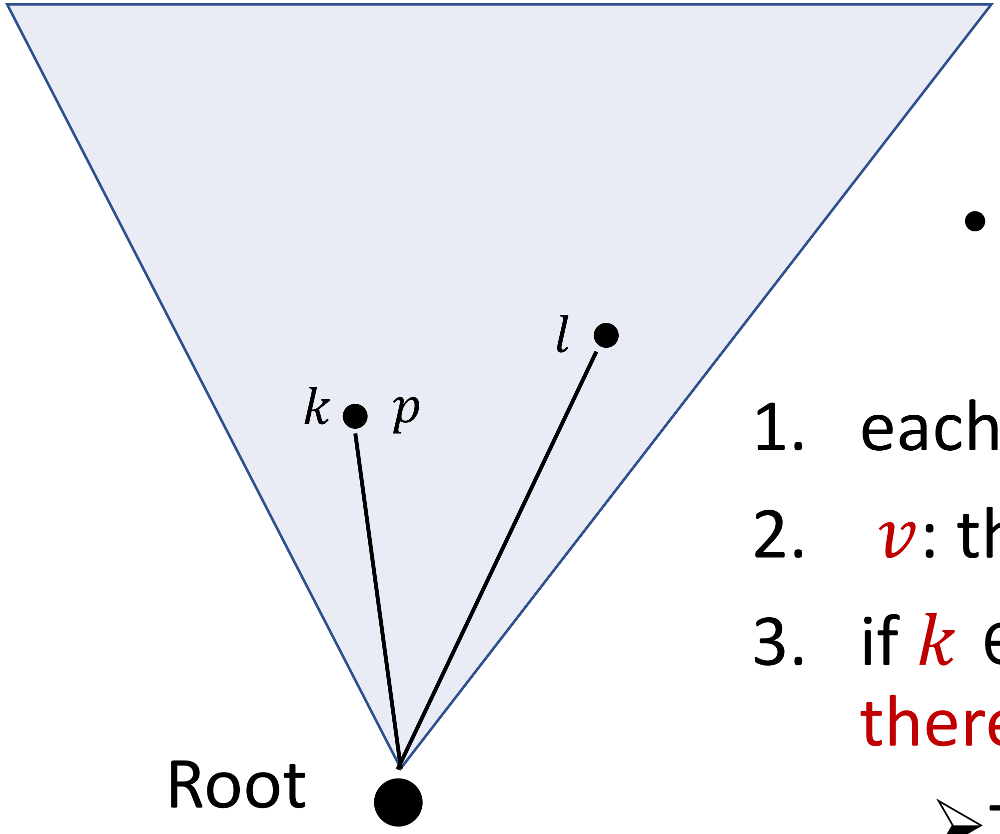


- "If satisfiable, then prevalent".
  - Maybe unnatural;
    - Collapses the two notions.
  - Wright rejected.
- But equivalent to (formalised by):
  - $\neg\neg A \to A$.
  - $(A \to \mathrm{B}) \to ((A \to \neg B) \to \neg A)$.
    - *Wright expected*.

Study of Prevalence

- The *object prevalence*: for all $d \in D$, $\vDash^{P} E(\overline{d})$.
- $\mathcal{W}_{\mathrm{P}}$: the class of the models with the formula prevalence & the object prevalence.
  - ➢ Formalisation of the relation with intuitionism (propositional case).

Models: Propositional Case



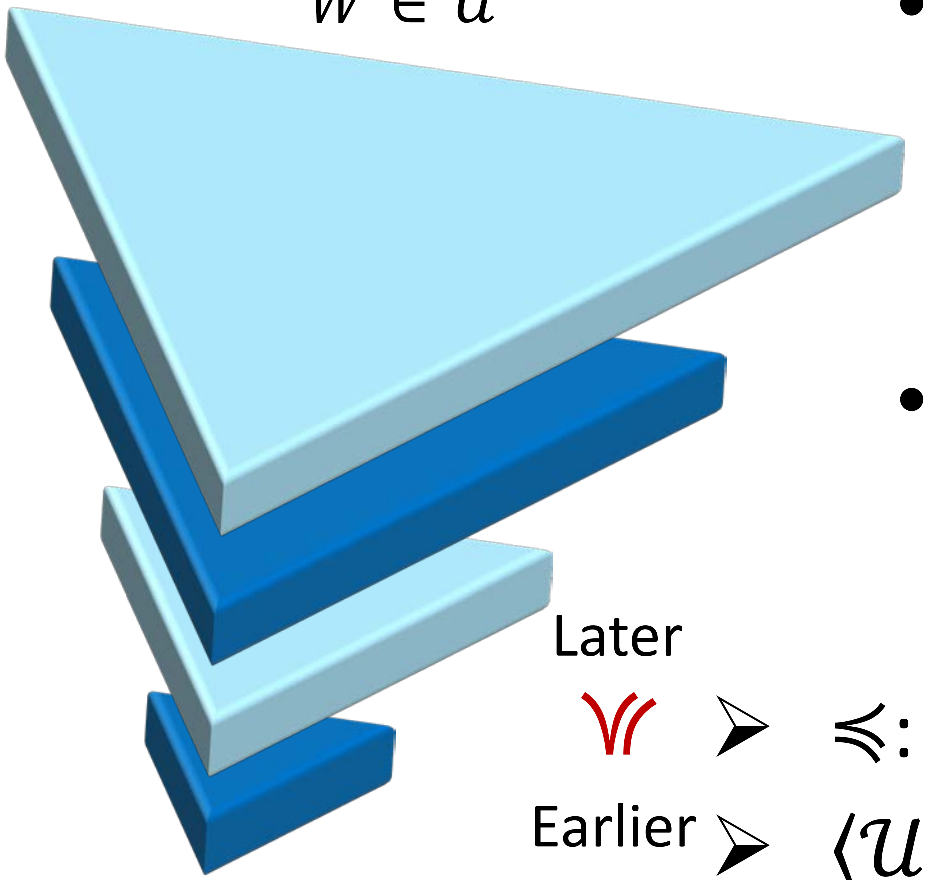
- Rooted tree-like intuitionistic models $\langle K, \leq, v \rangle$ such that…

1. each branch is at most countably long;
2. v: the valuation function.
3. if $k \in v(p)$ for some k, then for any $l \in K$, there is an $l' \geq l$ such that $l' \in v(p)$.
   - ➢The atomic prevalence condition.

➢Implies the formula prevalence of all complex formulas.

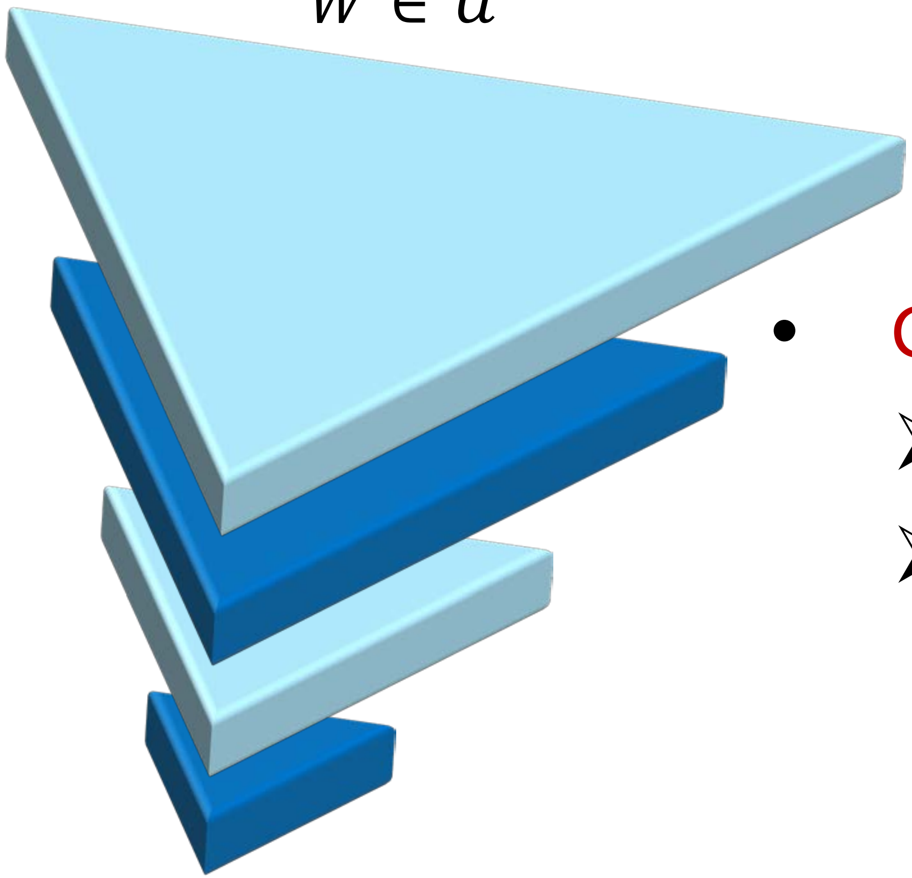
Ideas

One **SF** model = one **IPC** node



- Let $\mathcal{U} \subseteq \mathcal{W}_{\mathrm{P}}$, and arrange $\mathcal{U}$ in ascending order $\preccurlyeq$ of practical verification power. Then $\langle \mathcal{U}, \preccurlyeq \rangle$ is an **IPC** model. $\{p \in \mathrm{Var} | \vDash^{P} p\}$.
- $\langle \mathcal{U}, \preccurlyeq \rangle$ represents the same agent from various generations with increasing power.
  - $\preccurlyeq$: "generation order".
  - $\langle \mathcal{U}, \preccurlyeq \rangle$: "generation (g-)structure".
  - $\mathcal{G}$: the class of all g-structures.

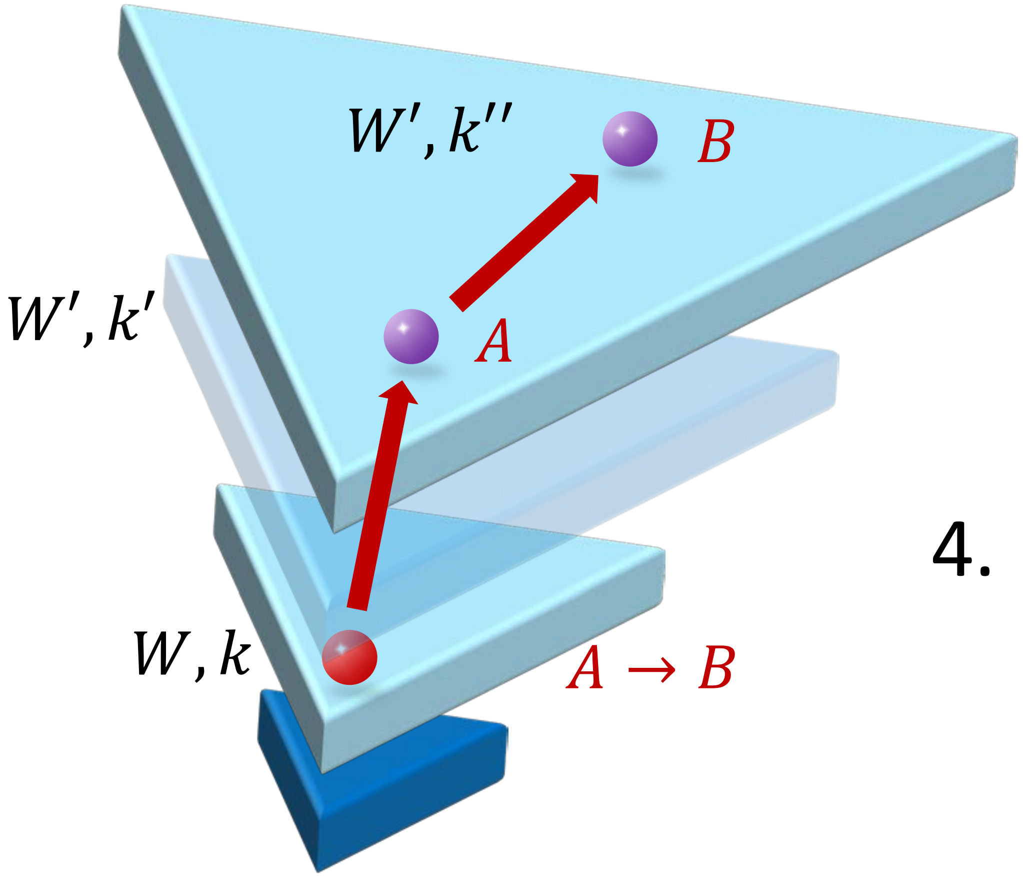
A New Relation: G-Forcing $\Vdash$



- $G = \langle \mathcal{U}, \preccurlyeq \rangle \in \mathcal{G}$.
- $W = \langle K_W, \leq_W, v_W \rangle \in \mathcal{U}$.

- G-valuation $v_G : \mathrm{Var} \to \mathcal{P}(\mathcal{U} \times \bigcup_{W \in \mathcal{U}} K_W)$.
  - $(W, k) \in v_G(p)$ iff $k \in v_W(p)$.
  - Then v_G persists.

A New Relation: G-Forcing ⊩



- $G = \langle \mathcal{U}, \preccurlyeq \rangle \in \mathcal{G}$.
- $W = \langle K_W, \leq_W, v_W \rangle \in \mathcal{U}$.

- G-forcing $\Vdash_G$:
  1. $W, k \Vdash_G p$ iff $(W, k) \in v_G(p)$;

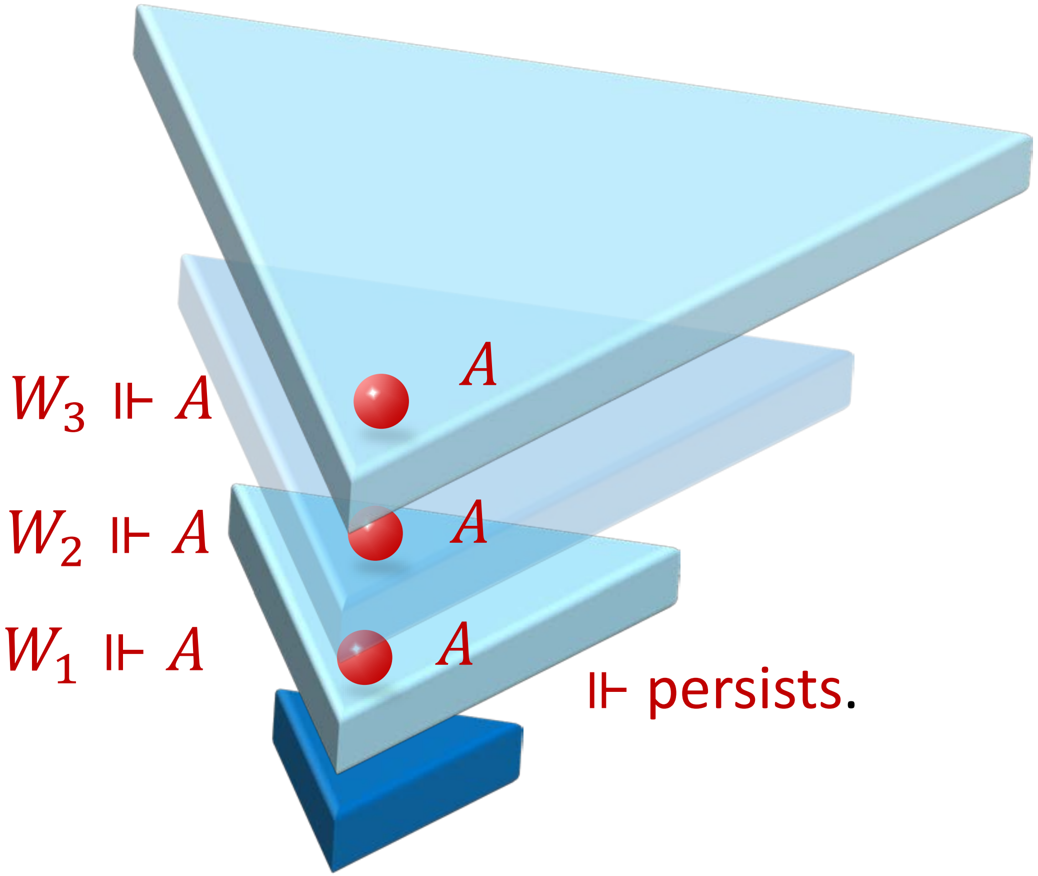
     …

  4. $W, k \Vdash_G A \to B$ iff for any $W' \succcurlyeq W$ and $k'(\in K_{W'}) \geq k$, if $W', k' \Vdash_G A$, then for some $k''(\in K_{W'}) \geq k'$, $W', k'' \Vdash_G B$.

- ⊩ is an SF forcing with increasing power.

Theorem

$\mathcal{G}$ to IPC



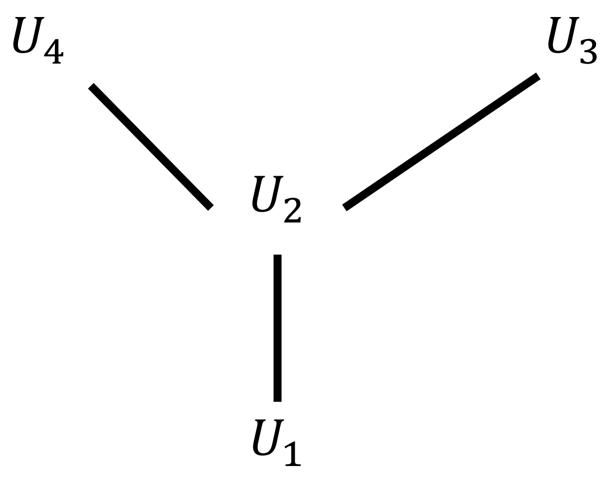
- $G = \langle \mathcal{U}, \preccurlyeq \rangle \in \mathcal{G}$.
- $W = \langle K_W, \leq_W, v_W \rangle \in \mathcal{U}$.

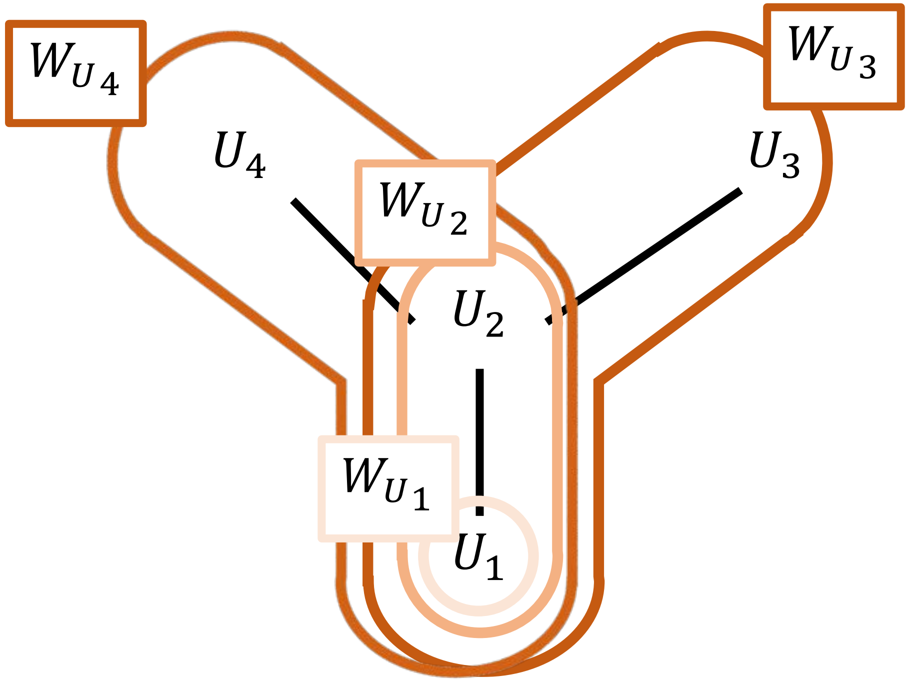
- Define an IPC valuation v: Var $\to \mathcal{U}$:
  - $W \in v(p)$ iff $\vDash_W^P p$ (iff $k \vDash p$ for some $k \in K_W$).
  - "$\Vdash$" captures practical verifiability in each W.
- $I_G = \langle \mathcal{U}, \preccurlyeq, v \rangle$ is an **IPC** model.

- $W \Vdash_{I_G} A$ iff $W, k \Vvdash_G A$ for some $k \in K_W$.

IPC to $\mathcal{G}$

- Any **IPC** model $I = \langle U^*, \preccurlyeq^*, v^* \rangle$ always induces a g-structure G_I…





such that for all $U \in U^*$, $U \Vdash A$ iff $W_U, U \Vvdash A$ in G_I.

“In Principle” = “in Practice” + Extension

- A is *valid in* $G \in \mathcal{G}$ if $W, k \Vvdash_G$ for all pair (W, k).
- Then A is valid in all $G \in \mathcal{G}$ iff $\vdash_{\mathbf{IPC}} A$.
  - Formalises: “verifiability in principle” is “verifiability in practice with extension of verification power” in this sense.

Summary

- Reconstructed & presented a formalisation of Crispin Wright's *strict finitistic logic*.
    - ➢ Reproduced the (decidability) principle.
    - ➢ Revealed the rejected principle "prevalence" is equivalent to something expected.
    - ➢ Formalised "verifiability in principle = verifiability in practice + extension in power" in the prevalent class.

A. Unused Slides

Lineage of Strict Finitistic Ideas

- 1982: Wright, “Strict Finitism”
- 1975: Dummett, “Wang’s Paradox”
- 1958: Yessenin-Volpin
- 1935: Bernays

Intuitionism — In principle

Strict Finitism — In practice

Classical Equivalences

- Prevalence is classical: in any $W \in \mathcal{W}_{\mathrm{P}}$,
  - $\vDash^{P} A \wedge B$ iff $\vDash^{P} A$ and $\vDash^{P} B$.
  - $\vDash^{P} A \vee B$ iff $\vDash^{P} A$ or $\vDash^{P} B$.
  - $\vDash^{P} A \rightarrow B$ iff $\nvDash^{P} A$ or $\vDash^{P} B$.
  - $\vDash^{P} \neg A$ iff $\nvDash^{P} A$ iff $\vDash^{P} \sim A$
  - $\vDash^{P} \forall x A$ iff $\vDash^{P} A(d)$ for all $d \in \mathrm{D}$.
  - $\vDash^{P} \exists x A$ iff $\vDash^{P} A(d)$ for some $d \in D$.