There is no Minimal Essentially Undecidable Theory

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Joint Work

This lecture reports joint work with


Fedor Pakhomov and Juvenal Murwanashyaka.

See our ArXiv preprint.

The GR Environment

Reflection on the Gödel-Rosser Theorem leads to many questions.

GR provides independent sentences for wide class of theories. It is rather independent of things like strength. In these characteristics, it differs from *concrete* or *tangible* incompleteness.

One studies properties guaranteeing incompleteness and/or undecidability. Examples are: *essential undecidability, essential hereditary undecidability, recursive inseparability, effective inseparability.*

We zoom in on *essential undecidability*.

A consistent RE theory U is *essentially undecidable* if every consistent RE extension of it is undecidable (or equivalently: incomplete).

Two Kinds of Extension 1

We can can read U extends V as *U extends V in the same language* or as *U interprets V*.

We write $K : U \rhd V$ for K is an interpretation of V in U. This means that there is a translation of V in U that commutes with the propositional connectives and that commutes with the quantifiers modulo a number of further features . . .

Examples: the interpretation of arithmetic in set theory, the interpretation of the Hyperbolic Plane in the Euclidean Plane.

We write $V \rhd U$ iff, there is a K with $K : V \rhd U$.

The results we will be discussing are rather insensitive w.r.t. the precise details of the notion of interpretation that we consider. So, we will leave our description vague.

Two Kinds of Extension 2

Theorem

Suppose every extension-in-the-same-language of V is undecidable and $U \rhd V$. Then U is undecidable.

Proof.

Suppose $K : U \rhd V$. Let $V' := \{\varphi \mid U \vdash \varphi^K\}$. Then, V' extends V. Moreover, if U were decidable, then so would V'. *Ergo*, U is undecidable. ❑

So we can define essential undecidability equivalently either using extensions-in-the-same-language or interpretation-extensions.

Example: R

The Tarski-Mostowski-Robinson theory R is given by:

R1. $\vdash \underline{m} + \underline{n} = \underline{m+n}$

R2. $\vdash \underline{m} \cdot \underline{n} = \underline{m \cdot n}$

R3. $\vdash \underline{m} \neq \underline{n}$, for $m \neq n$

R4. $\vdash x \leq \underline{n} \rightarrow \bigvee_{i \leq n} x = \underline{i}$

R5. $\vdash x \leq \underline{n} \vee \underline{n} \leq x$

R is the primary example of an essentially undecidable base theory.

We can drop R5, but not R4, since RCF, the theory of real closed fields, is decidable. (One can show that School := R1,2,3 and R1,2,3,5 are both undecidable, even if they have RCF as extensions.)

R has more salient properties in this connection: *essentially hereditarily* undecidable and *effectively inseparable*.

The Question

How nice it would be when a theory like R would be the best choice for an essentially undecidable base theory.

For R, we can find many incomparable ones and, also, weaker ones.

But, perhaps, there is another interpretability minimal essentially undecidable theory?

Alas, as we will show, this is not the case.

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Supremum and Infimum

$U \varovee V$ is a theory in the disjoint sum of the signatures of U and V plus a fresh 0-ary predicate symbol P. The theory is axiomatised by all $P \to \varphi$, where φ is an axiom of U plus $\neg P \to \psi$, where ψ is an axiom of V.

We have $(U \varovee V) \rhd W$ iff $U \rhd W$ and $V \rhd W$. So, $U \varovee V$ is *the interpretability infimum* of U and V.

Suppose U and V are essentially undecidable and W is a consistent extension of $U \varovee V$. Then, either $W + P$ or $W + \neg P$ is consistent. Say $W + P$ is consistent. The theory $W + P$ is a finite extension of U. So, $W + P$ must be undecidable. But, then, W must be undecidable (since decidability is preserved to finite extensions). Similarly, in case $W + \neg P$ is consistent.

So: if there is an interpretability *minimal* essentially undecidable theory, then there must be an interpretability *minimum*.

Overview

Shoenfield's Argument 1

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{Z}$ range over RE sets of numbers. We confuse every RE set with one of its indices.

Theorem

For every $\mathcal{A}$ we can effectively find disjoint $\mathcal{B}, \mathcal{C} \leq_{\mathrm{T}} A$, such that for every $\mathcal{D}$ that separates $\mathcal{B}$ and $\mathcal{C}$, we have $\mathcal{A} \leq_{\mathrm{T}} \mathcal{D}$.

Proof: We define:

- $x \in \mathcal{Z}$ iff $\exists z\, \mathrm{T}_1((x)_1, x, z)$.
- $x \in \mathcal{B}$ iff $((x)_0 \in \mathcal{A}) < (x \in Z)$.
- $x \in \mathcal{C}$ iff $(x)_0 \in \mathcal{A} \wedge x \in \mathcal{B}^{\perp}$.

It is immediate that $\mathcal{B}, \mathcal{C} \leq_T \mathcal{A}$ and that $\mathcal{B} \cap \mathcal{C} = \emptyset$.

Suppose $\mathcal{D}$ separates $\mathcal{B}$ and $\mathcal{C}$. We show that $\mathcal{A} \leq_{\mathrm{T}} \mathcal{D}$. Let d be an index of $\mathcal{D}$. We note that $\langle w, d\rangle \in \mathcal{Z}$ iff $\langle w, d\rangle \in \mathcal{D}$.

Shoenfield's Argument 2

We first show: $w \in \mathcal{A}$ iff $\langle w, d\rangle \in \mathcal{B}$. Right to left is immediate. Suppose $w \in \mathcal{A}$. Then either $\langle w, d\rangle \in \mathcal{B}$ or $\langle w, d\rangle \in \mathcal{C}$. In case $\langle w, d\rangle \in \mathcal{C}$, we find $\langle w, d\rangle \in \mathcal{Z}$ and, hence $\langle w, d\rangle \in \mathcal{D}$. $\lightning$ So, $\langle w, d\rangle \in \mathcal{B}$.

If $\langle w, d\rangle \notin \mathcal{D}$, then $\langle w, d\rangle \notin \mathcal{B}$, so $w \notin \mathcal{A}$.

If $\langle w, d\rangle \in \mathcal{D}$, then $\langle w, d\rangle \in \mathcal{Z}$. So, we can effectively determine whether $\langle w, d\rangle \in \mathcal{B}$ and, thus, whether $w \in \mathcal{A}$. ❑

Overview

Janiczak-Shoenfield Theories 1

Let JS be the theory of one equivalence relation plus, for every n:

a. There are at least n equivalence classes with at least n elements.
b. There is at most one equivalence class with precisely n elements.

Theorem

Every sentence over JS *is equivalent to a boolean combination of sentences* A_n *stating that there is an equivalence class of precisely n elements.*

JS is recursively boolean isomorphic to propositional logic. By a result of Kripke and Pour-El, all effectively inseparable RE theories are recursively boolean isomorphic to R. Moreover R is boolean isomorphic to propositional logic, but, of course, not recursively boolean isomorphic to propositional logic.

Turing Persistence

- T is *Turing persistent*, iff, for all $U \rhd T$, we have $U \geq_{\mathrm{T}} T$.

Suppose T is Turing persistent. Then, T is essentially undecidable iff T is undecidable.

There are Turing persistent and, hence, essentially undecidable RE theories in every RE Turing degree that are *not* recursively inseparable. I still have to check the argument in detail.

Shoenfield-Janiczak Theories

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be as in Shoenfield's Theorem. As before, we confuse RE sets with their indices.

Let sh($\mathcal{A}$) be JS plus A_n for $n \in \mathcal{B}$ and $\neg\,\mathsf{A}_n$ for $n \in \mathcal{C}$. Then, sh($\mathcal{A}$) is Turing equivalent to $\mathcal{A}$. It is Turing persistent and, hence, essentially undecidable iff not recursive.

Suppose $K : U \rhd \mathrm{sh}(\mathcal{A})$, then $\{n \mid U \vdash \mathsf{A}_n^K\}$ separates $\mathcal{B}$ and $\mathcal{C}$. Hence, $U \geq_{\mathrm{T}} \mathrm{sh}(\mathcal{A})$.

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Rogers and Mostowski

Let Rec be the set of indices of recursive sets.

Theorem (Rogers, Mostowski)

Rec *is complete* Σ^0_3.

The Proof

Proof.

Suppose $U^\star$ that is the interpretability minimum of the recursively enumerable essentially undecidable theories. We have:

$$\begin{array}{rll} \mathcal{A} \notin \text{Rec} & \text{iff} & \text{sh}(\mathcal{A}) \text{ is essentially undecidable} \\ & \text{iff} & \text{sh}(\mathcal{A}) \rhd U^\star \end{array}$$

Since, interpretability between recursively enumerable theories is Σ^0_3, it follows that Rec is Π^0_3. *Quod non*. ❑

We have an alternative more direct proof that does not use the Rogers-Mostowski result.

Thank You